Measurement of Inequality

by

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Abstract

The analysis of inequality is placed in the context of recent developments in economics and statistics. Prepared for *Handbook of Income Distribution* edited by A. B. Atkinson and F. Bourguignon.

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1 Introduction

1.1 Inequality and income distribution

Inequality measurement is a subject where a lot of energy can be spent arguing about the meaning of terms. This is not a matter of taxonomy for the sake of taxonomy. The problem is that "inequality" itself - as with many other economic concepts - is not self-defining and that the definitions applied may derive from sometimes sharply contrasted intellectual positions. Inequality measurement is an attempt to give meaning to comparisons of income distributions in terms of criteria which may be derived from ethical principles, appealing mathematical constructs or simple intuition. In this respect it is similar to other methods of characterising and comparing income distributions to which it is closely related. For this reason we shall take into account some of these other concepts - such as general principles of distributional ranking - rather than just concentrating upon inequality measurement in the narrow sense.[1]

Given that we are to focus upon comparisons of "income distributions" we should acknowledge straight away that this is itself a flexible term and in the present context may be interpreted broadly to apply to distributions of other economic entities that share some of the same analytical constructs or empirical characteristics. This remark refocuses our attention upon another basic issue: What is an income distribution?

There is a variety of stylised answers to this question. For the present exposition we will concentrate upon two useful paradigms.

1.1.1 The Irene and Janet approach

The essential idea can be presented in terms of a very simple two-person economy, as depicted in figure 1 - but this can easily be generalised in a number of ways.[2] Specifically, if one assumes that "income" conveys all that one might want to know about an individual's economic status, then the income distribution can be represented as a list of persons and a list of corresponding incomes: in the n-person version, if x_i denotes the income of person $i, i = 1, ..., n$ then a common approach to the issue is just to provide a corresponding ordered list of the incomes. In the simplest case the distribution is simply represented as a finite-dimensioned vector:

$$\mathbf{x} = (x_1, x_2, ..., x_n) \tag{1}$$

[1]For previous contributions that have surveyed the properties of inequality measures see Champernowne (1974), Cowell (1995), Foster (1985), Jenkins (1991), Lambert (1993), Sen (1973), Sen and Foster (1997). A good general introduction to the welfare-economic issues is also to be found in Atkinson (1983).

[2]This is the approach commonly adopted in the modern theoretical work on income inequality: see, for example, Dasgupta et al. (1973), Sen (1973).

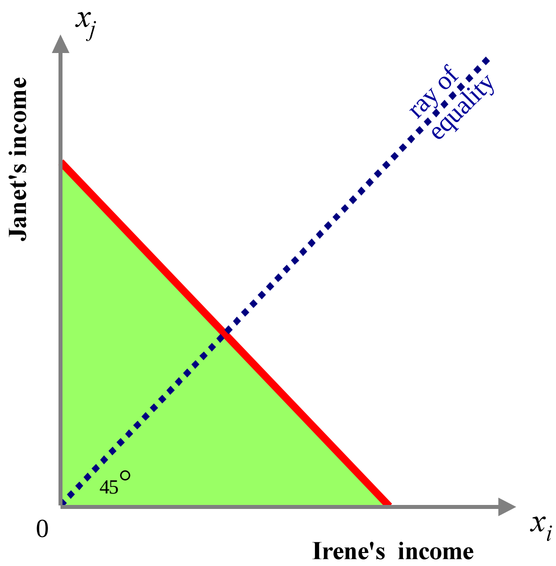

Figure 1: The Irene-and-Janet approach to income distribution

In the two-person case the set of all feasible income distributions out of a given total income is illustrated by the shaded area in Figure 1 bounded by a 45° line.[3] Other features of income distributions which it may be desirable to model can be fitted within this general framework. For example if different income-receiving units consist of families of differing size we might want to represent this by introducing a corresponding set of population weights for the observations, so that the distribution becomes an ordered list of pairs:

$$((w_1, x_1), (w_2, x_2), ..., (w_n, x_n)) \tag{2}$$

Clearly either the single variable case (1) or the multivariable case (2) can be applied to any situation which may be modelled as a known, finite set of individuals.

1.1.2 The Parade approach

Alternatively we could depict an income distribution using some aspect of the general statistical concept of a probability distribution. This is brilliantly captured by the famous story of the "parade of dwarfs and a few giants" related by Jan Pen (Pen 1971). The simple and compelling imagery of the parade - according to which each person's income is represented by his physical height - provides

[3]The shaded area (other than the 45° line) consists of distributions where some income is thrown away. The counterpart to this line in the case of an n-person economy this is an $(n-1)$-dimensional simplex.

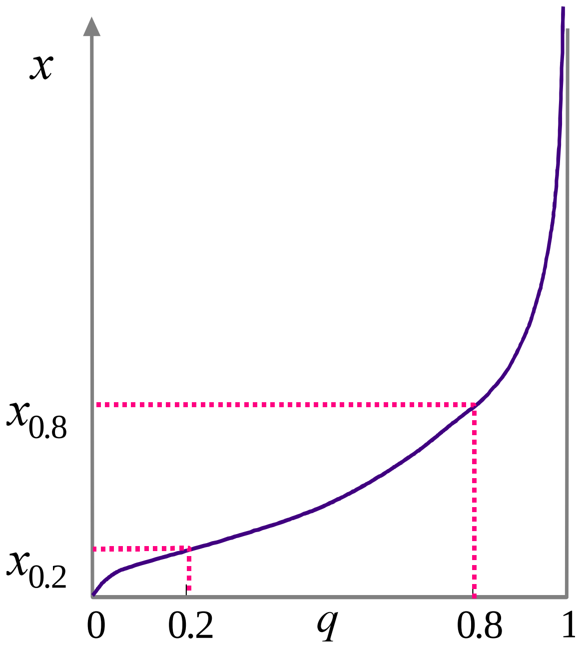

Figure 2: Pen's Parade

more than just an appealing parable for inequality in terms that a lay person can appreciate. It also suggests that welfare in a society can be expressed in the form of an "income profile" of members of the population. The idea is illustrated in Figure 2: along the horizontal axis we measure proportions of the population q, with income x along the vertical axis. The population is arranged in ascending order of income ("height") and the typical pattern shape of the resulting profile is illustrated by the solid curve in Figure 2: the points $x_{0.2}$ and $x_{0.8}$ give the income ("height") of the person who appears exactly twenty and eighty percent of the way along the Parade respectively.

One reason that this is so useful is that immediately we can interpret ideas in inequality analysis in terms of analogous statistical concepts. By adopting the fiction that there may be uncountably many income-receivers in the population we obtain a device that will permit simple interpretation and implementation, and that is applicable to both discrete and continuous distributions: the Parade in Figure 2 is a simple transformation of the statistical distribution function F - see Figure 3 in subsection 2.2 below. Trading on this important relationship we shall call this general approach the "F-form".

Which to use? Of course the two approaches are reconcilable. In effect they are just alternative simplifying representations of an inherently complex subject. For reasons both of economic principle and statistical tractability it is reasonable to apply each of the two approaches to different types of problems in distributional analysis. For example, the F-form approach can be especially useful in cases where it is appropriate to adopt a parametric model of income distribution and inequality; the Irene and Janet paradigm can be particularly convenient in approaches to the subject based primarily upon individualistic welfare criteria, where a simplified, discrete representation of an income distribution is often appropriate. Because of the generality of the F-form[4] we shall mainly use this in the discussion which follows.

These two analytical threads have been around for some time, as a brief glance at the history of the subject reveals: the line of argument pursued by Pigou and Dalton (Pigou 1912), (Dalton 1920), who may be claimed to have laid the basis for the welfare-theoretic approach to inequality, is based on the model-free "Irene-and-Janet" approach; by contrast Pareto's insights on inequality comparisons were almost quintessentially those of a model-based approach to inequality (Pareto 1896); the seminal work of Gini and Lorenz (Gini 1912), (Lorenz 1905) although originally formulated in F-form terminology avoided the restrictions of the Paretian parametric approach and today their insights are commonly reinterpreted in terms of the "Irene-and-Janet" paradigm.

1.2 Overview

Perhaps it is appropriate to say a word or two about what this chapter will not do. Because the primary focus is upon inequality within a static framework, broader issues of social welfare and poverty get a brief look-in, mobility and polarisation almost none. Furthermore, although we shall consider issues of empirical implementation, there is no coverage of actual examples: one has to draw the line somewhere.

We shall tackle the inequality-measurement problem within the general class of questions concerning distributional analysis. The order of attack will be as follows. Section 2 examines fundamental issues involved in representing the problem of inequality comparisons, and addresses questions of how we may represent an income distribution and the basis upon which distributional comparisons are to be made. The fundamental techniques of comparison in section 3 lead to powerful and implementable criteria for ranking distributions. However, general ranking criteria applied to income distributions - whether based upon *ad hoc* methods or formal welfare economics - very often result in indecisive comparisons. For some purposes it is desirable to have a unique inequality index, and we will consider the axiomatic and welfare-theoretic approaches to this in sections 4 and 5. A number of ramifications and extensions of the basic analysis including the

[4]Note that I do not claim it as more general than the Irene-Janet approach: it requires the use of a probability measure which will ensure that the anonymity axiom and population principle (see the discussion on page 9 below) are automatically satisfied - see, for example Hoffman-Jø rgensen (1994a), page 100.

structure of inequality and multidimensional considerations and approaches are examined in sections 6 and 7. Finally section 8 addresses problems of empirical implementation.

2 Distributional judgements

A serious approach to inequality measurement should begin with a consideration of the entities to which the tools of distributional judgment are being applied: what is being distributed amongst whom? Many of these issues are well-rehearsed in the literature and so we will only give a cursory treatment here touching on the points that bear directly upon inequality analysis.

2.1 Income and the individual

A coherent definition of "equality" in this context requires implementable definitions of income and of the income recipient. However, each of these concepts raises difficulties for the theoretician and practical analyst which should not just be brushed aside.

Even if we set aside the important theoretical difficulties associated with the definition of individual well-being, and the observation, measurement and valuation of assets,[5] there is an obvious gap in the meaning of "income": between an abstraction that represents "individual welfare", and a mundane practical concept such as "total family income" which may be dictated by accounting conventions. The standard approach to the bridging of this gap is to introduce an *equivalence scale* which defines a "rate of exchange" between conventionally-defined income y and an adjusted concept of income x - equivalised income - which acts as a money metric of utility. Imagine that a complete description of a family or household's circumstances other than money income can be given by some list of attributes $\mathbf{a}$ (age of each family member, health indicators,...) then we suppose that there is some functional relationship χ such that[6]

$$x = \chi(\mathbf{a}, y) \tag{3}$$

This relationship is usually expressed in the form

$$x = \frac{y}{\nu(\mathbf{a})}, \tag{4}$$

where $\nu(.)$ is a function determining the number of *equivalent adults*. There is of course a range of difficulties associated with a specification such as (4): for

[5] For a discussion of the problems of valuing incomes see Fisher (1956), and on the issues raised by looking at the distribution of income rather than that of ability see Allingham (1972).

[6] The use of equivalised incomes can have major - and sometimes apparently bizarre - impacts on distributional comparisons (Glewwe 1991).

example it is not clear what the appropriate analytical basis for the function χ in (3) should be, nor even why there should be a proportional relationship between x and y.[7] An alternative approach to the modelling of needs is discussed in section 7 below.

Again the concept of "income recipient" is sometimes treated as though it were self-defining, when in practical application of inequality comparisons this is manifestly not the case. Given that the structure of conventional welfare economics is essentially based on the concept of the *individual person* whereas data on income distribution is very often collected on a *family* or *household* basis some transformation of income-recipient - logically separate from the equivalisation process (3) - is required for meaningful distributional comparisons to be made. In sum, if a dataset consists of household attribute-income pairs $(\mathbf{a}_i, y_i)$ then, in order to adduce the income distribution that is relevant according to individualistic welfare criteria, the standard approach requires that the incomes x_i in (2) should be the equivalised incomes found from a relationship such as (3) and the weights w_i in (2) should correspond to the number of persons in each household (Cowell 1984, Danziger and Taussig 1979), although alternative coherent views have been persuasively argued.[8]

2.2 Distributional concepts

We tackle the problem by introducing an abstract notation for the distribution that will encompass both the elementary Irene-and-Janet approach and also other important cases. It will also facilitate the development of the statistical approach to the analysis of income distributions.[9] Let $\mathfrak{F}$ be the space of all univariate probability distributions with support $\mathfrak{X} \subseteq \Re$, where $\Re$ denotes the set of real numbers and $\mathfrak{X}$ is a proper interval. We may use $\mathfrak{F}$ as the basis for modelling income distribution: $x \in \mathfrak{X}$ is then a particular value of income and $F \in \mathfrak{F}$ is one possible distribution of income in the population; so $F(x_0)$ captures the proportion of the population with income less than or equal to some value x_0 as in Figure 3. The set $\mathfrak{X}$ is important, because it incorporates an implicit assumption about the logically possible values that x could adopt: in practice it will be determined by the precise economic definition of "income" - see 8.1.1 below. In addition we will write $\underline{x} := \inf(\mathfrak{X})$, and we use $\mathfrak{F}(\mu)$ for the subset of $\mathfrak{F}$ with given

[7] One these issues see Coulter et al. (1992a) and Cowell (1999a). The ethical issues associated with equivalisation are considered in Sen (1998) section 7 and the issues of estimation and implementation in Danziger and Jäntti (1998) section 2.4 and Gottschalk and Smeeding (1998).

[8] See for example Ebert (1995c, 1997d), Pyatt (1990); see also discussion of this issue in Bruno and Habib (1976), Ebert (1995a). The logic of using the family as a basic economic unit in this context is discussed in Bottiroli Civardi and Martinetti Chiappero (1995).

[9] See Section 8 below. Note that adoption of this analogy does not of course imply that an individual's income is stochastic. The analysis of inequality where incomes are stochastic - in particular the problems of reconciling *ex-ante* and *ex-post* concepts of inequality is addressed in Ben-Porath et al. (1997).

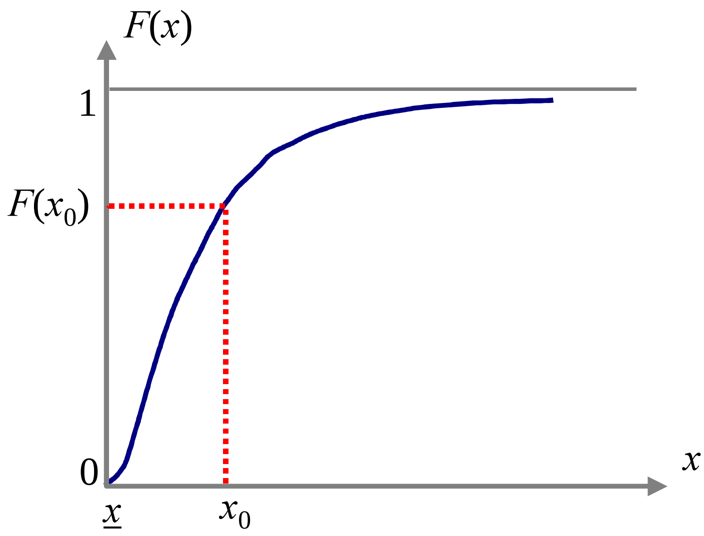

Figure 3: An income distribution

mean μ: we need this for the many cases in inequality measurement where we want to consider distributions out of a fixed-size "cake". This basic framework can be extended to handle multivariate distributions (discussed further in section 7 below) by introducing the corresponding space of r-dimensional probability distributions $\mathfrak{F}_r$ (so that $\mathfrak{F}_1 = \mathfrak{F}$).

The function F is our fundamental concept for economic and statistical approaches to the subject and represents a formalisation of the F-form concept of the income distribution introduced above (note that Figure 3 is the inverse of - i.e. a simple rotation and reflection of - Figure 2). The standard summary statistics of distributions can easily be expressed in terms of this concept: for example the mean is a functional $\mu : \mathfrak{F} \mapsto \Re$ given by

$$\mu(F) := \int x dF(x) \tag{5}$$

Furthermore, using the concept of the F-function we can conveniently capture a very wide range of theoretical and empirical distributions, including some important special cases.

For example, if $F \in \mathfrak{F}$ is absolutely continuous over some interval $\mathfrak{X}' \subseteq \mathfrak{X}$ then we may also define the *density function* $f : \mathfrak{X}' \mapsto \Re$ - see Figure 4; if F is also differentiable[10] over $x \in \mathfrak{X}'$ then f is given by

$$f(x) := \frac{dF(x)}{dx}. \tag{6}$$

[10]Note that this is not a necessary requirement for f to exist: for example the density could be positive everywhere, but discontinuous at some points.

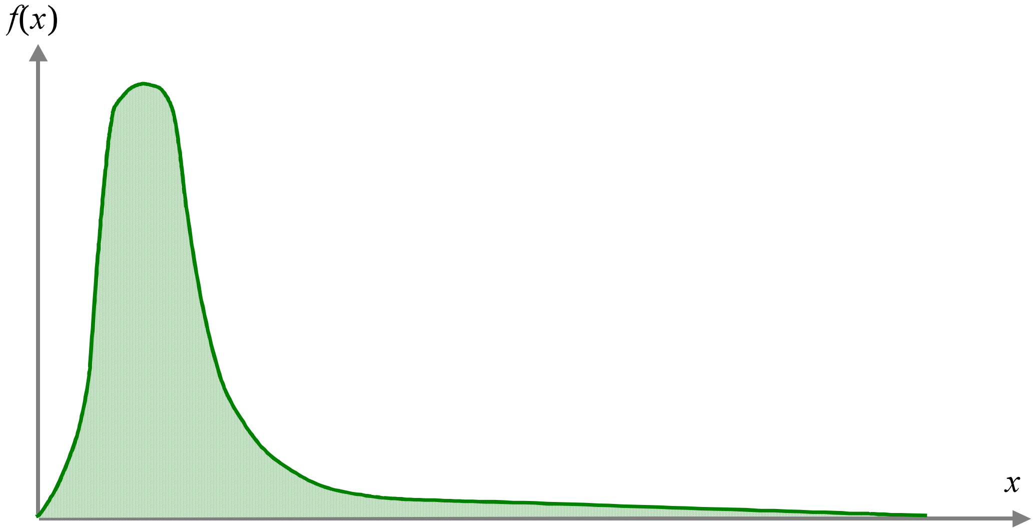

Figure 4: The density function

In some cases it is easier and more intuitive to work with f rather than with the corresponding F. On the other hand the framework is sufficiently flexible to deal with cases where F is not differentiable. For example, the elementary representation (1) can be expressed as:

$$F(x) = \frac{j}{n} \text{ if } x \geq x_{[j]} \tag{7}$$

where $x_{[j]}$ represents the jth smallest component of (1).[11]

This basic framework for distributional analysis can be applied not only to inequality measurement, but also to other related issues such as social-welfare and poverty comparisons. Each of these separate issues can be illuminated by considering the analytical linkages amongst them (Cowell 1988b), (Foster and Shorrocks 1988a, 1988b).

[11] If $x_1, x_2, ..., x_n$ are all distinct we can write (7) in the slightly more transparent form:

$$dF(x) = \begin{cases} \frac{1}{n} & \text{if } x = x_1 \\ \dots & ... \\ \frac{1}{n} & \text{if } x = x_n \\ 0 & \text{otherwise} \end{cases} .$$

2.3 Distributional and welfare axioms

Let us consider the key concepts that we use to compare distributions in the context of inequality measurement. First, let us use the term *inequality ordering* to mean a complete and transitive binary relation $\succcurlyeq_I$ on $\mathfrak{F}$;[12] if this ordering is continuous it can be represented as a functional $I : \mathfrak{F} \mapsto \Re$. Other distributional concepts to which inequality may be related can be similarly expressed. For example a *social-welfare ordering* of distributions will be written $\succcurlyeq_W$; equivalently a *social-welfare function* (SWF) can be expressed in the form of a functional $W : \mathfrak{F} \mapsto \Re$. In each case the strict ordering part $\succ$ and the equivalence part $\sim$ will be defined in the usual way, and the properties of $\succcurlyeq_I$ and $\succcurlyeq_W$ will be determined by ethical principles or fundamental distributional axioms:[13] a brief overview of some of the standard axioms will be useful.

In our analytical framework the first two basic assumptions that we need to make may be expressed in terms of an elementary vector-representation of a distribution (1). The first is very straightforward:

- *Anonymity.*[14]

$$(x_1, x_2, x_3, ..., x_n) \sim_I (x_2, x_1, x_3, ..., x_n) \sim_I (x_1, x_3, x_2, ..., x_n), ...$$

This assumption - which is usually invoked for welfare orderings $\succcurlyeq_W$ also - states that all permutations of personal labels are regarded as distributionally equivalent. It requires that the ordering principle use only the information about the income variable and not about, for example, some other characteristic which might be discernible in a sample or an enumeration of the population. However the axiom is neither trivial nor self-evident, and under certain circumstances for specific problems of distributional analysis it could make sense to relax it or to modify its scope of application. For example suppose one has information on a variety of income attributes of individuals: perhaps there is sufficient detail about personal circumstances to infer the bivariate income distribution $F(x_t, x_{t-1})$ where x_t is current income and x_{t-1} is income last period; then an analysis of the distribution of current income (only) that invokes the anonymity axiom is making the very strong assumption that $\succcurlyeq_I$ or $\succcurlyeq_W$ does not take account

[12] This means $\forall F, G \in \mathfrak{F}$ either or both of the statements "$F \succcurlyeq_I G$", "$G \succcurlyeq_I F$" are true and, $\forall F, G, K \in \mathfrak{F}$, "$F \succcurlyeq_I G$" and "$G \succcurlyeq_I K$" together imply "$F \succcurlyeq_I K$". See, for example, the definition of strict order Fishburn (1970), page 11 and ordering Suzumura (1983), page 7 and, in the inequality context, the complete pre-ordering of Fields and Fei (1978). The representation of such an ordering by a continuous function follows from the classic work of Debreu (1954). However representation results for some restricted types of orderings on subsets of the space of distributions can run into difficulties - see Wakker (1993)

[13] See also definition 1 below. The relationship between inequality and social welfare is discussed more fully in Section 5 below.

[14] Also known as symmetry.

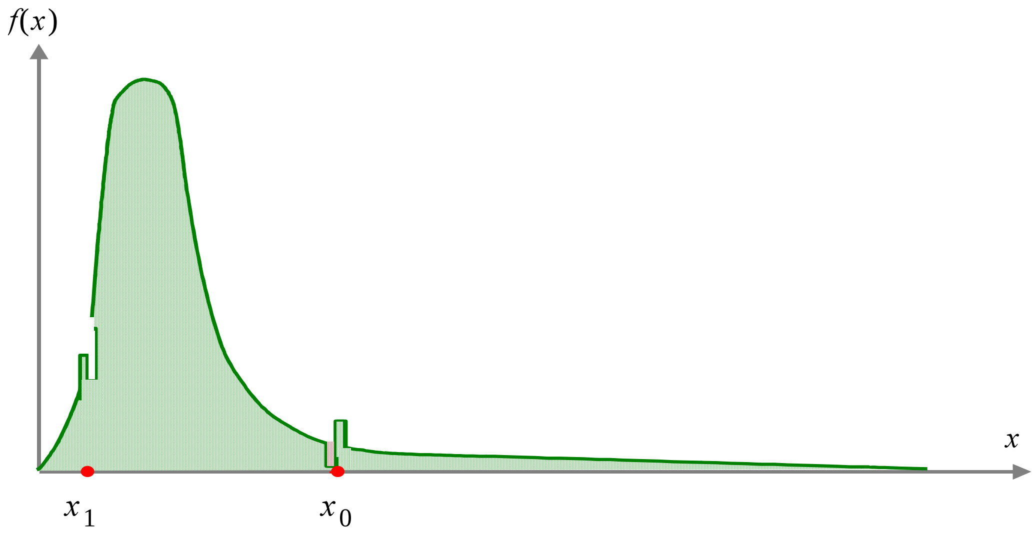

Figure 5: A mean-preserving spread

of the past (the marginal distribution of x_{t-1}) nor of the links between the past and the present (for example the correlation between x_t and x_{t-1}).

Similar considerations apply in situations where individual utility is presumed to depend both on income and on some other attribute which cannot be aggregated into the individual income concept.[15] In what follows we shall assume that the distributional problem has been sufficiently well defined to make questioning of the anonymity principle unnecessary.

- *The population principle.*(Dalton 1920)

$$\begin{aligned}(x_1, x_2, ..., x_n) \; &\sim_I \; (x_1, x_1, x_2, x_2, ..., x_n, x_n) \sim_I \; ... \\ &\sim_I \; (\underbrace{x_1, \ldots, x_1}_{m}, \underbrace{x_2, \ldots, x_2}_{m}, ..., \underbrace{x_n, \ldots, x_n}_{m}) \; ...\end{aligned}$$

The population principle states that an income distribution is to be regarded as distributionally equivalent to a distribution formed by replications of it. Once again there may be reasons for querying this principle under certain circumstances[16] but we shall not pursue them here.

[15] In this type of case it may be appropriate to apply a more general version of the anonymity principle to the problem of multi-dimensional distributional comparisons. See, for example, Atkinson and Bourguignon (1982, 1987), Cowell (1985b), Lambert and Yitzhaki (1995).

[16] See for example Cowell (1995) page 56. This axiom is sometimes invoked also for welfare and poverty comparisons. The consistency of inequality comparisons across distributions with differing populations is discussed in Salas (1998).

These two axioms permit us to work with elementary distributions represented in the F-form. In particular they permit us to define an *equal* distribution, given that the concept of income and of income-receiver have been settled; this is the degenerate distribution $H^{(x*)}$ given by:

$$H^{(x*)}(x) = \begin{cases} 1 & \text{if } x \geq x^* \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

which places a single point mass at x^*. But, of course, by themselves these two axioms do not get us very far. The following principle is usually taken to be indispensable in most of the inequality literature.[17].

- *Principle of transfers.* (Dalton 1920)(Pigou 1912) $G \succ_I F$ if distribution G can be obtained from F by a *mean-preserving spread* .[18]

The idea of a mean-preserving spread is illustrated in Figure 5 depicting two equal and opposite deformations of the income distribution at points x_0 and x_1. In the context of the Irene-and-Janet approach to distribution this principle can be represented thus: consider an arbitrary distribution $\mathbf{x}_A := (x_1, ..., x_i, ..., x_j, ..., x_n)$ and a number δ such that $0 < \delta < x_i \leq x_j$; from $\mathbf{x}_A$ we may form the distribution $\mathbf{x}_B := (x_1, ..., x_i - \delta, ..., x_j + \delta,\ ..., x_n)$. The principle of transfers then ranks $\mathbf{x}_B$ as more unequal than $\mathbf{x}_A$.[19]

In addition to these principles a large number of theoretical and empirical studies explicitly invoke additional axioms which may be motivated by principles of economic welfare or considerations of structure within the space of income distributions. The principal welfare axiom that bears upon distributional rankings may be expressed as:

- *Monotonicity.* $G \succ_W F$ if distribution G can be obtained from F by a rightward translation of some probability mass - see Figure 6 in which this translation takes place at point x_0. In the Irene-and-Janet approach this can be represented thus: consider an arbitrary distribution $\mathbf{x}_A := (x_1, ..., x_i, ..., x_n)$ and a number $\delta > 0$; from $\mathbf{x}_A$ we form the distribution $\mathbf{x}_B := (x_1, ...x_i + \delta, ..., x_n)$. The monotonicity principle then requires that welfare is higher in $\mathbf{x}_B$ than $\mathbf{x}_A$.[20]

[17] See, for example, Atkinson (1970), Atkinson (1983), Cowell (1995), Sen (1973).

[18] This is the way the axiom would be expressed if the ordering criterion were to be defined so as to correspond with an economic "bad" like inequality or poverty. For a "good", like social welfare, one simply reverses the sign: $F \succ_W G$ in the above definition. Note that Figure 5 depicts a case where the two points of deformation are on opposite sides of the mean; of course the twin perturbations could occur on the same side of the mean. Note also that Dalton (1920) refined the concept of the transfer principle which was originally set out in Pigou (1912) - see Amiel and Cowell (1998); Castagnoli and Muliere (1990) give a broader interpretation of the

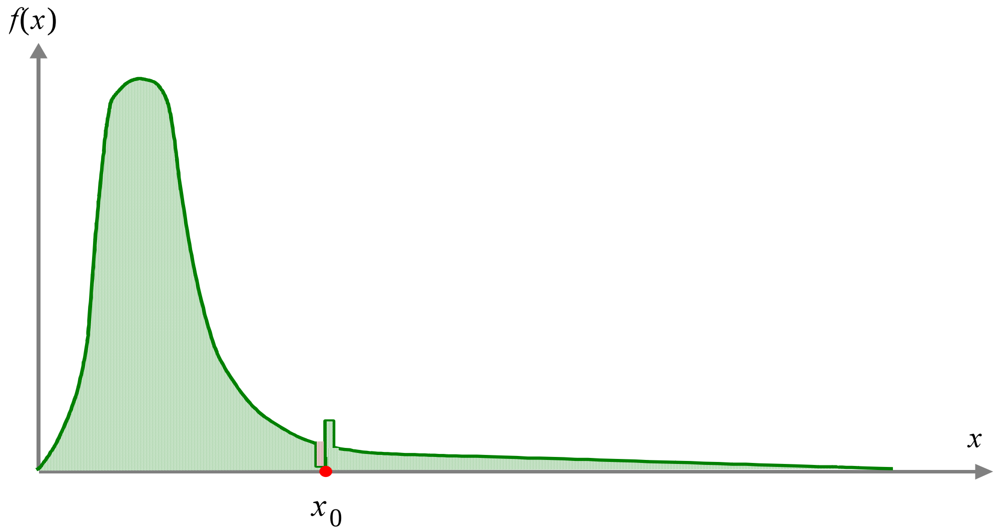

Figure 6: Income growth at x_0

To introduce the principal structure axioms let $F^{(+k)}$ be the distribution derived from F by a shift or translation by an amount $k \in \Re$:

$$F^{(+k)}(x) = F(x - k) \tag{9}$$

Likewise let $F^{(\times k)}$ be the distribution derived from F by transforming the income variable by a scalar multiple $k \in \Re_+$:

$$F^{(\times k)}(x) = F\left(\frac{x}{k}\right) \tag{10}$$

The following structural axioms are stated in terms of inequality, but could equally be applied also to welfare or poverty orderings.[21]

- *Scale Invariance.* Given $F, G \in \mathfrak{F}(\mu)$ if $G \succcurlyeq_I F$ then $G^{(\times k)} \succcurlyeq_I F^{(\times k)}$.

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transfer principle.

[19]Except for their ith and jth components the vectors $\mathbf{x}_A$ and $\mathbf{x}_B$ are identical.

[20]This axiom is commonly invoked also in the case of poverty orderings. Notice that it is similar, but not identical, to the Pareto criterion. The Pareto criterion is defined in terms of utilities rather than incomes, and will differ from monotonicity if individual utility functions are dependent on other people's incomes, as is reasonable in cases involving distributional judgments (Amiel and Cowell 1994c).

[21]The terminology for the following concepts is not uniform throughout the literature. The terms "invariance" and "independence" are variously defined, and the particular interpretations of scale- and translation- invariance used here are often described as homotheticity and translatability and are to be distinguished from the corresponding concepts of independence introduced in 5.3.4 below.

- *Decomposability.* Given $F, G, K \in \mathfrak{F}(\mu)$ and $\delta \in [0,1]$ then $G \succcurlyeq_I F$ implies $[1-\delta]\,G + \delta K \succcurlyeq_I [1-\delta]\,F + \delta K$.

This means that if the same distribution K is mixed with F and with G (where F, G, K all have the same mean) then ordering of the resulting mixture distribution is determined solely by the ordering of F and G - see also the discussion in Section 6 below.

The above list of axioms constitutes a brief summary of the standard approach to inequality measurement and associated welfare theory, and in Sections 3 to 5 we will see the role these play in making distributional comparisons and in determining an inequality index. However it should be noted that in many cases reasonable alternative approaches are available. For example it could be argued that monotonicity axiom is unacceptably strong; as an alternative we might require no more than the condition that welfare should increase if there were a uniform rightward translation of the whole distribution:[22]

- *Uniform income growth.* $k > 0 \Rightarrow F^{(+k)} \succ_W F$.

Again, in place of the scale-invariance concept it is sometimes argued that the following structural assumption is appropriate Kolm (1976a, 1976b):

- *Translation Invariance.* Given $F, G \in \mathfrak{F}(\mu)$ if $G \succcurlyeq_I F$ then $G^{(+k)} \succcurlyeq_I F^{(+k)}$.

In contrast to the standard axiom - where the inequality-contour map remains invariant under scalar transformations of income - this assumption ensures that the inequality-contour map remains invariant under uniform additions to income and under uniform subtractions from income; furthermore "intermediate" versions of invariance can be specified (Bossert 1988b) (Bossert and Pfingsten 1990) Kolm (1969, 1976a, 1976b).

Furthermore other coherent approaches to inequality can be developed that do not assume the individualistic structure that is implied by acceptance of the transfer principle; for example alternatives may based upon the concept of income differences - see Gastwirth (1974b), Kolm (1993), Temkin (1986, 1993). These alternative approaches raise the question of what constitutes an "appropriate" axiom system for distributional comparisons. This issue has been investigated by questionnaire-experimental testing[23] which has revealed that fundamental concepts such as the transfer principle do not correspond with the way in which people appear to make inequality comparisons in practice.

[22]Note that monotonicity implies the uniform income growth but not *vice versa*.

[23]See, for example, Amiel (1999), Amiel and Cowell (1992, 1994b, 1998, 1999a, 1999b), Ballano and Ruiz-Castillo (1992), Cowell (1985a), Harrison and Seidl (1994a, 1994b), Beckman et al. (1994); see also Kolm (1997c) Considerations of perceptions of income distribution also underlie recent interest in the topic of "polarisation" (Wolfson 1994).

3 Ranking distributions

The basic concept in the comparison of income distributions is that of a *ranking* over the set of distributions $\mathfrak{F}$. This is more general than that of an ordering - as conventionally used in the analysis of individual preferences, for example - in that the idea encompasses "partial orderings" as well as orderings. Use the notation $\succeq_T$ to indicate the ranking that is induced by some comparison principle T. Then we need to distinguish three possibilities in any distributional comparison:[24]

Definition 1 *For all $F, G \in \mathfrak{F}$:*
(a) (strict dominance) $G \succ_T F \Leftrightarrow G \succeq_T F$ and $F \not\succeq_T G$
(b) (equivalence) $G \sim_T F \Leftrightarrow G \succeq_T F$ and $F \succeq_T G$
(c) (non-comparability) $G \perp_T F \Leftrightarrow G \not\succeq_T F$ and $F \not\succeq_T G$.

3.1 Formal and informal approaches

It is common practice in empirical studies to use informal easily computable ranking criteria; this typically takes the form of quantile rankings or distributional-shares rankings of income distributions. The use of these tools has a direct intuitive appeal: statements such as "the differential between the top decile and the bottom decile has narrowed" and "the share of the bottom ten percent has risen", seem to be sensible ways of talking about inequality-reducing distributional changes. Furthermore these basic ideas are related to other intuitive concepts in distributional analysis. For example the *range*, $(x_{\max} - x_{\min})$, is sometimes used as an elementary - if extreme - inequality index, but the implementation of the range in practice may be as $(x_{0.99} - x_{0.01})$, for example.[25]

However it is also possible to give rigorous theoretical support to these intuitive approaches. To do this we use the concept of the social-welfare function, introduced on page 9. In particular we focus upon a special class of SWF, those

[24]Contrast this with the definition of an ordering given in note 2.3.

[25]For example, in Rawls' work on a theory of justice there is a discussion of how to implement his famous "difference principle" which focuses upon the least advantaged. Rawls himself suggests that it might be interpreted relative to a particular quantile of the distribution (the median) - see Rawls (1972) page 98. A number of useful pragmatic indices involving quantiles have been proposed such as the semi-decile ratio (Wiles 1974), (Wiles and Markowski 1971) and the comparative function of Esberger and Malmquist (1972).

that can be expressed in *additively separable* form; these are given by:[26]

$$W(F) = \int u(x) dF(x). \tag{11}$$

where $u : \mathfrak{X} \mapsto \Re$ is an *evaluation function* of individual incomes. Use the term $\mathfrak{W}_1$ for the subclass of SWFs of type (11) where u is increasing;[27] and use $\mathfrak{W}_2$ to denote the subclass of $\mathfrak{W}_1$ where u is also concave.

The SWF subclasses $\mathfrak{W}_1$ and $\mathfrak{W}_2$ will be found to play a crucial role in interpreting two fundamental ranking principles - first- and second-order distributional dominance[28] - and to have a close relationship with the intuitively appealing concepts of quantiles and shares.

3.2 First-order distributional dominance.

First-order dominance criteria are based on the quantiles of the distribution which are yielded by the (generalised) inverse of the distribution function F. Let us make this more precise:

Definition 2 *For all* $F \in \mathfrak{F}$ *and for all* $0 \leq q \leq 1$, *the* quantile functional *is defined by:*[29]

$$Q(F; q) = \inf\{x | F(x) \geq q\} = x_q \tag{12}$$

[26]Or a monotonic transformation of (11). The additively separable structure of W is quite a strong requirement. For example if welfare and inequality are linked by a relation such as (32) below then additively separability of W is stronger than the decomposability axiom of the inequality ordering (an additively separable W implies decomposability of $\succcurlyeq_I$but not vice versa). Also note that (11) has the interpretation of "expected utility" where a representative person in the population regards it as equally likely that he should have any of the income-entitlements in the income distribution (Harsanyi 1955). For general discussion see Broome (1991), Kolm (1996b, 1996a), Roemer (1996), Zajac (1996).

[27]This specification is consistent with the assumption of the Pareto principle and the absence of externalities in the SWF - see Amiel and Cowell (1994c) for a discussion of the issues involved here.

[28]This terminology is inherited from the stochastic dominance literature. (Bawa 1975)

[29]See Gastwirth (1971). On the other hand the quantiles can be defined in the form of a correspondence. In this case if the function F is not continuous the quantile may be a *set* of income values. Defining the collection of subsets of $\mathfrak{X}$:

$$\Xi := \{\{x : a \leq x \leq b\} : a, b \in \mathfrak{X}\}$$

The quantile correspondence is

$$\tilde{Q} : \mathfrak{F} \times [0, 1] \mapsto \Xi$$

such that

$$\tilde{Q}(F; q) = \{x : F(x) = q\}$$

- Cf. Kendall and Stuart (1977), pp. 39-41.

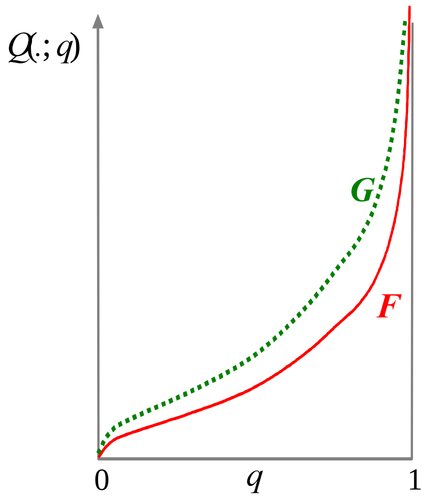

Figure 7: G first-order dominates F

For example $Q(F; 0.1)$ is the first decile of the distribution F, and $Q(F; 0.5)$ is the median of F. For any distribution of income F, the graph of Q describes, in formal terms, the concept of the Parade introduced in 1.1.2 above. This concept of the profile implies that if some persons "grow" (and nobody shrinks) social welfare also increases. In formal terms we may express these ideas by means of the following theorem (Quirk and Saposnik 1962), (Saposnik 1981, 1983):

Theorem 1 $G \succeq_Q F$ *if and only if,* $W(G) \geq W(F)\ \forall (W \in \mathfrak{W}_1)$.[30]

This result means that the quantiles contain important information about economic welfare. If each quantile in distribution G is no less than the corresponding quantile in distribution F , and at least one quantile is strictly greater (as in Figure 7) then distribution G will be assigned a higher welfare level by every SWF in class $\mathfrak{W}_1$.

3.3 Second-order distributional dominance

Unfortunately the first-order criterion - the ranking-principle $\succeq_Q$ - has a couple of drawbacks. First, in practical applications, it is very often the case that neither distribution first-order dominates the other. Second, it does not employ all the standard principles of social welfare analysis: above all it does not incorporate the principle of transfers. For this reason it is useful to introduce the second-order

[30]See definition 1.

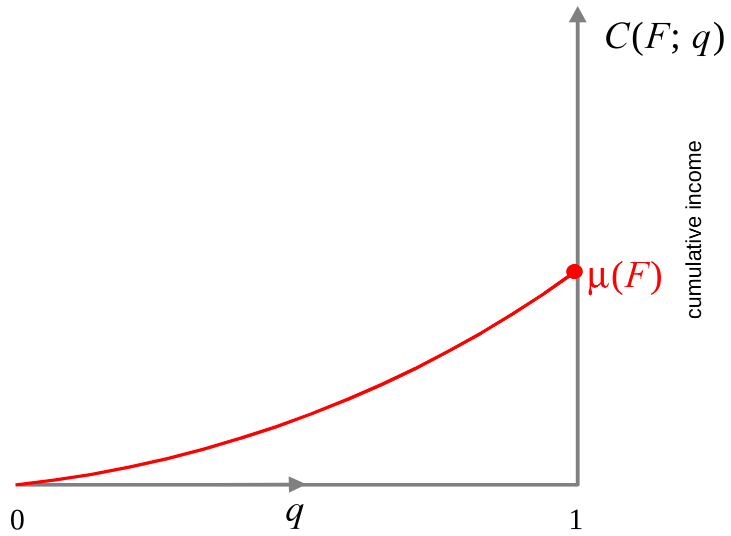

Figure 8: The income-cumulation diagram

dominance criterion.[31] The application of the second-order dominance criterion requires the following concept:

Definition 3 *For all* $F \in \mathfrak{F}$ *and for all* $0 \leq q \leq 1$, *the* cumulative income functional *is defined by:*

$$C(F; q) := \int_{\underline{x}}^{Q(F;q)} x dF(x). \tag{13}$$

Note that, by definition $C(F; 0) = 0$, $C(F; 1) = \mu(F)$, and that, for a given $F \in \mathfrak{F}$, the graph of $C(F, q)$ against q describes the *generalised Lorenz curve* (GLC) - see Figure 8.[32]

The principal property of C can be summarised in the following theorem:[33]

Theorem 2 $\forall F, G \in \mathfrak{F}$: $G \succeq_C F$ *if, and only if,* $W(G) \geq W(F)$ $\forall (W \in \mathfrak{W}_2)$.

[31] However Bishop et al. (1991) argue that in international comparisons the second-order criterion $\succeq_C$ in Theorem 2 below does not resolve many of the "incomparable cases" where $G \perp_Q F$.

[32] This terminology is not universal: Kolm (1969) refers to the graph of $C(F, q)$ as the "concentration curve" and Yitzhaki and Olkin (1991) uses the term "absolute Lorenz curve".

[33] See Kakwani (1984), Kolm (1969), Marshall and Olkin (1979), Shorrocks (1983). For more on GLC rankings and the relationship with first-order criteria see Iritani and Kuga (1983), Thistle (1989a, 1989b). For discussion of the use of the GLC as a general criterion for ranking stochastic income streams see Saposnik and Tutterow (1992).

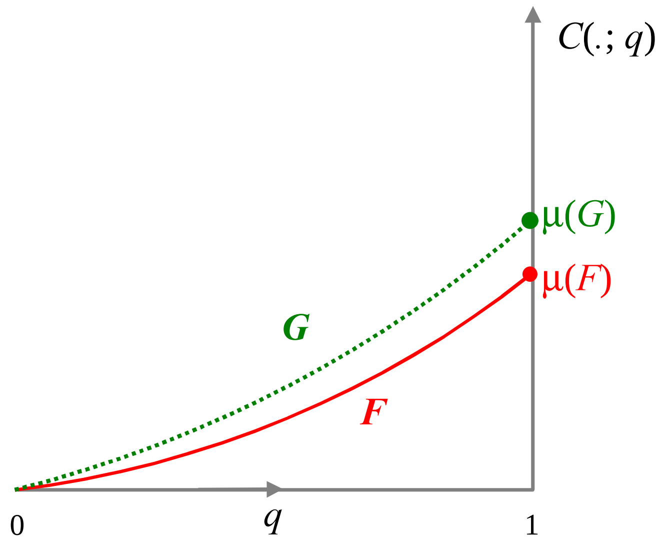

Figure 9: G second-order dominates F

3.4 Tools for income distribution

The GLC is a fundamental tool for drawing conclusions about welfare from individual income data. Closely associated with it are other important tools of distributional analysis. Foremost among these is the conventional Lorenz curve (Lorenz 1905) - or *relative Lorenz curve* to distinguish it from other concepts with similar names.

3.4.1 The Lorenz ranking

To obtain the Lorenz ranking we normalise the cumulative income functional by the mean:

$$L(F;q) := \frac{C(F;q)}{\mu(F)} \tag{14}$$

The Lorenz curve - the graph of $L(F;q)$ against q[34] - encapsulates the intuitive principle of the distributional-shares ranking referred to above: in Figure 10 it is evident that the income share of the bottom $100q\%$ of the population must be higher in distribution G than in F, whatever the value of q.

The basic insights of Theorem 2 were originally obtained for distributions with a given mean $\mathfrak{F}(\mu)$:[35]

[34]For a given F the first moment function $\Phi : \mathfrak{X} \mapsto [0,1]$ is simply $\Phi(x) = L(F;F(x)) = \frac{1}{\mu(F)} \int_{\underline{x}}^{x} y dF(y)$ - (Kendall and Stuart 1977). The general use of moment functions in measuring inequality is discussed in Butler and McDonald (1987).

[35]The principal reference is the seminal paper of Atkinson (1970) whose work was inspired

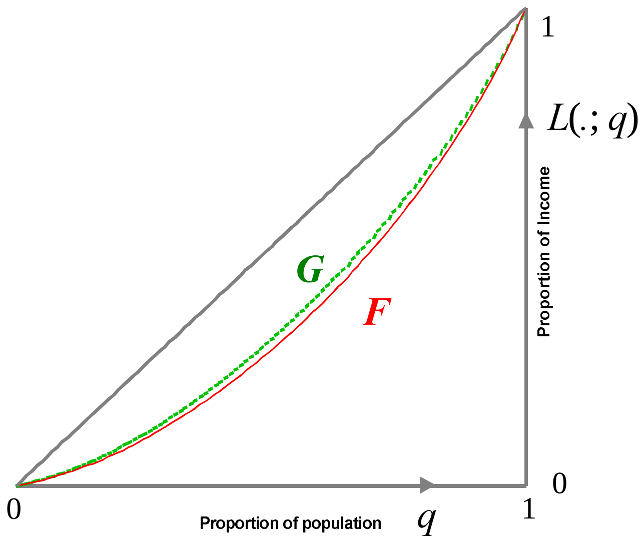

Figure 10: G (relative) Lorenz-dominates F

Theorem 3 *$\forall F, G \in \mathfrak{F}(\mu)$: $G \succeq_L F$ if, and only if, $W(G) \geq W(F)$ $\forall(W \in \mathfrak{W}_2)$.*

3.4.2 Relative and absolute dominance

An alternative similar reinterpretation of Theorem 2 can be obtained by restricting the admissible SWFs to those in $\mathfrak{W}_2$ that have the additional property that proportional increases in all incomes yield welfare improvements:

$$\left\{W \left| W \in \mathfrak{W}_2; \forall F \in \mathfrak{F}, k > 1 : W\left(F^{(\times k)}\right) > W(F)\right.\right\} \tag{15}$$

Distribution G dominates F for SWFs in this restricted class if and only if G (relative) Lorenz-dominates F and $\mu(G) \geq \mu(F)$. Other special cases of Theorem 2 also yield useful insights. In particular consider the welfare property analogous to (15) that uniform absolute increases in all incomes yield welfare improvements:[36]

$$\left\{W \left| W \in \mathfrak{W}_2; \forall F \in \mathfrak{F}, k > 0 : W\left(F^{(+k)}\right) > W(F)\right.\right\}. \tag{16}$$

In this case the counterpart to (14) is the *absolute Lorenz curve* (ALC) (Moyes 1987):

---

by results in the stochastic dominance literature. However, based on the work of Hardy et al. (1934), Dasgupta et al. (1973) showed that the class of SWFs in Theorem 3 can be broadened to those that are S-concave but not necessarily additively separable. See also Arnold (1987), Fields and Fei (1978), Kolm (1966, 1968, 1969), Kurabayashi and Yatsuka (1977), Rothschild and Stiglitz (1973).

[36]This is another way of stating the principle of uniform income growth (Champernowne and Cowell 1997) - see page 13 - and is sometimes known as the incremental improvement condition (Chakravarty 1990).

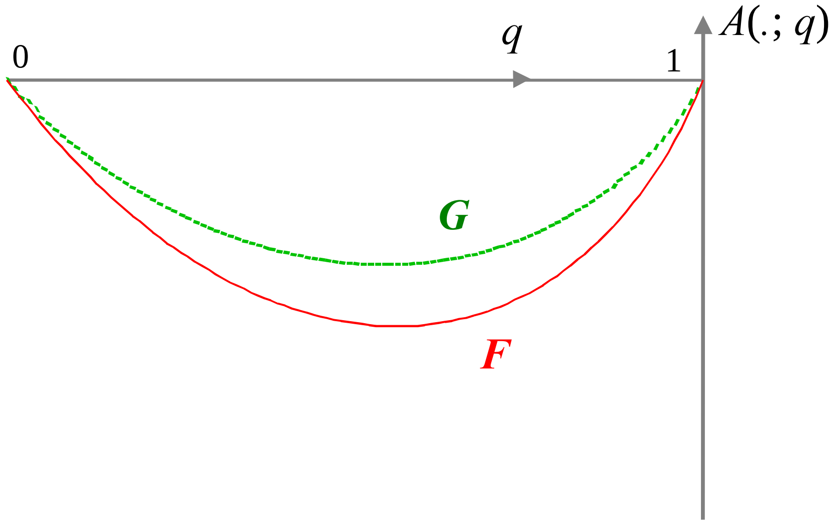

Figure 11: G absolute Lorenz-dominates F

$$A(F;q) := C(F;q) - q\mu(F) \tag{17}$$

Then we find that $G \succeq_A F$ (see Figure 11) and $\mu(G) \geq \mu(F)$ if, and only if, $W(G) \geq W(F)$ for all W that satisfy (16) (Shorrocks 1983). The ALC is a particularly convenient tool for comparing distributions where a large proportion of the incomes are negative.[37]

3.4.3 Extensions

However, just as with the first-order criterion, one may often find in practice that second-order criteria are indecisive. In situations where Lorenz curves intersect there are essentially two routes forward.[38] The first is to supplement the restrictions on the class of SWFs (11) - which means imposing a further restriction on the income-evaluation function; the second approach - discussed in section 4 - is to derive specific unambiguous indices of inequality. There is no shortage of additional restrictions that could reasonably be imposed upon the $\mathfrak{W}$-classes: one of the more useful is the so-called "principle of diminishing transfers"

[37]If $\mu(F)$ is positive then the presence of negative incomes causes no problem for the relative Lorenz curve; but if there are so many negative incomes that $\mu(F) \leq 0$ then it is clear from (14) that there may be problems (Amiel et al. 1996).

[38]Basu (1987), following Sen (1973), has argued that inequality comparisons are inhererently imprecise and that rather than seeking to make progress with incomplete partial orderings $\succeq$, one should approach the subject using the concept of a fuzzy binary relation. See also Ok (1995, 1996).

(Kolm 1976a) - namely that a small transfer from an individual with income x to one with income $x - \Delta$ (where Δ is some given absolute dollar amount) should have a greater impact on inequality the lower x is located in the distribution.[39] If this additional principle is invoked then a result is available for some cases where the Lorenz curves intersect.[40] This result is closely linked to a concept of "third-order" dominance (Shorrocks and Foster 1987); an extension of the idea of dominance to an arbitrary order is discussed in Fishburn and Willig (1984) and Kolm (1974, 1976b) and Formby, Smith, and Zheng (1996) discuss the topic of "normalised" dominance - essentially adapting nth-order dominance to $\mathfrak{F}(1)$, the subset of $\mathfrak{F}$ with mean unity.

4 An axiomatic approach to inequality measurement

As we have noted in section 3.4, it is in the nature of general ranking principles that in many practical situations they yield an "indecisive" answer: "$F \perp G$".This is one reason that it is often considered desirable to go beyond the use of principles in order to derive practically implementable indices: specific examples of the inequality-measure concept introduced on page 2.3: The ways in which this step is to be done can be categorised roughly into three types of approaches:

- An *ad hoc* selection procedure for methods that may have a neat statistical or graphical interpretation.[41]
- The axiomatic approach which invites the "user" to specify what the basic principles are for comparing distributions; sufficiently tightly-specified principles may narrow down the range of tools to a small number of indices.
- The welfare-theoretic approach in which an explicit SWF is adopted as a basis for distributional judgment; an inequality measure may then be inferred from the specified SWF.

[39]The principle is implied by the principle of "transfer sensitivity" (Shorrocks and Foster 1987) which also goes by the name of "aversion to downside inequality" (Davies and Hoy 1995), the latter based upon Menezes et al. (1980).

[40]Specifically $W(F) > W(G)$ for all SWFs in this restricted class if $F, G \in \mathfrak{F}$ have all three of the following properties (i) $\mu(F) = \mu(G)$, (ii) $\text{var}(F) \leq \text{var}(G)$, and (iii) $\exists q^* \in (0, 1)$ such that $\forall q < q^* : L(F; q) > L(G; q)$ and $\forall q > q^* : L(F; q) < L(G; q)$ - a single Lorenz intersection result (Atkinson 1973), (Davies and Hoy 1994a), (Davies and Hoy 1994b), (Dardanoni and Lambert 1988), (Muliere and Scarsini 1989). Davies and Hoy (1995) extend the analysis to cases of multiple Lorenz intersections. Zoli (1998) discusses an extension of this to a "positionalist" interpretation of the principle of diminishing transfers - see Mehran (1976) and section 4.2 below. See also Subramanian (1987).

[41]See, for example, the elegant and appealing interpretation of the Gini index described on page 25 below.

These three categories are by no means mutually exclusive. Rather they are complementary routes to a set of useful and implementable indices. It is appropriate to consider first the *ad hoc* and axiomatic approaches (there is little point in trying to separate them: it is nearly always possible to find some set of axioms to support the use of a particular *ad hoc* measure that happens to have intuitive appeal[42]) in order to see how properties of inequality indices can be linked to fundamental ideas about the meaning of inequality comparisons. The SWF route is considered in more detail in section 5.

In all three types of approach we need the following basic concepts:

Definition 4 *Two inequality indices* $I, \hat{I} : \mathfrak{F} \mapsto \Re$ *are* ordinally equivalent *if there is a function* $\psi : \Re^2 \mapsto \Re$*, increasing in its second argument, such that:*

$$\forall F \in \mathfrak{F} : \hat{I}(F) = \psi\left(\mu(F), I(F)\right) \tag{18}$$

Definition 5 *An inequality index* $I : \mathfrak{F} \mapsto \Re$ *is* zero-normalised *if* $I(H^{(\mu)}) = 0$.[43]

Definition 6 *Two zero-normalised ordinally equivalent inequality indices* $I, \hat{I} : \mathfrak{F} \mapsto \Re$ *are* cardinally equivalent *if the function* ψ *in (18) is linear in its second argument.*

Notice that the ordinal-equivalence relation embodied in the function ψ in (18) may depend on the mean of the distribution: this dependence will give rise to some problems of interpretation in section 5.3.

4.1 Insights from information theory

To obtain insights on the nature of income distribution comparisons one might reasonably look to the analysis of distributions in other fields of study. Using an analogy with the entropy concept in information theory Theil (1967) pioneered an approach to inequality measurement from which a number of lessons may be drawn for the axiomatic approach to inequality measurement.

[42] For a discussion of the issues here see Foster (1994).

[43] This normalisation - which means that inequality is zero for a distribution where everyone has the same income - can always be trivially ensured: for any measure I the zero-normalised index I^* is where $I^*(F) := I(F) - I\left(H^{(\mu(F))}\right)$. I and I^* will order all distributions in $\mathfrak{F}(\mu)$ in the same way. Sometimes great store is laid on normalisation such that maximum inequality is standardised at 1. Although this is of no great analytical significance it can always be arranged in the sense that, for a given I, an ordinally equivalent I^* can be found that is bounded in $[0,1]$. Furthermore if an inequality measure is bounded $[0,1]$. there are infinitely many ordinally-equivalent indices that are also bounded in $[0,1]$.

4.1.1 The Theil approach

The information-theoretic idea incorporates the following main components (Kullback 1959):

- A set of possible events each with a given probability of its occurrence.
- An information function ϕ for evaluating events according to their associated probabilities, similar in spirit to the income-evaluation function u in (11). The calibration of ϕ uses three key axioms: (1) if an event was considered to be a certainty ($p = 1$) the information that it had occurred would be valueless ($\phi(1) = 0$); (2) higher-probability events have a lower value ($p > p' \Rightarrow \phi(p) < \phi(p')$); (3) the joint information of two independent events is the sum of the information of each event separately ($\phi(pp') = \phi(p) + \phi(p')$). These requirements ensure that the evaluation function is $\phi(p) = -\log(p)$.
- The entropy concept is the expected information in the distribution.

Theil's application of this to income distribution replaced the concept of event-probabilities by income shares, and introduced:

- A comparison distribution, usually taken to be perfect equality.

Given some appropriate normalisation this approach then found expression in the following inequality index (Theil 1967):

$$I_{\text{Theil}}(F) := \int \frac{x}{\mu(F)} \log\left(\frac{x}{\mu(F)}\right) dF(x) \tag{19}$$

and also the following (which has since become more widely known as the *mean logarithmic deviation*):

$$I_{\text{MLD}}(F) := -\int \log\left(\frac{x}{\mu(F)}\right) dF(x) \tag{20}$$

4.1.2 A generalisation

However, in their original derivation, the Theil measures in 4.1.1 use an axiom (#3 in the abbreviated list above) which does not make much sense in the context of distributional shares. It has become common practice to see (19) and (20) as two important special cases of a more flexible general class; in terms of the Theil analogy this is achieved by taking a more general evaluation function for income shares.

Then the *generalised entropy* (GE) family of measures (Cowell 1977) (Cowell and Kuga 1981a, 1981b) (Toyoda 1975) is given by:

$$I_{\mathrm{GE}}^{\alpha}(F) := \frac{1}{\alpha^2 - \alpha} \int \left[\left[\frac{x}{\mu(F)} \right]^{\alpha} - 1 \right] dF(x) \tag{21}$$

where $\alpha \in (-\infty, +\infty)$ is a parameter that captures the sensitivity of a specific GE index to particular parts of the distribution: for α large and positive the index is sensitive to changes in the distribution that affect the upper tail; for α negative the index is sensitive to changes in the distribution that affect the lower tail.[44] Measures ordinally equivalent to the GE class include a number of pragmatic indices such as the variance and the coefficient of variation

$$I_{\mathrm{CV}}(F) := \sqrt{\int \left[\frac{x}{\mu(F)} - 1 \right]^2 dF(x)}, \tag{22}$$

standard statistical moments (Kendall and Stuart 1977), and measures of industrial concentration(Gehrig 1988), (Hannah and Kay 1977), (Hart 1971), (Herfindahl 1950).

However the principal attraction of the GE class (21) lies neither in the generalisation of Theil's insights, nor in the happy coincidence of its connection with well-known indices, but rather in the fact that the class embodies some of the key distributional assumptions which we discussed in section 2.3.

Theorem 4 *A continuous inequality measure $I : \mathfrak{F} \mapsto \Re$ satisfies the principle of transfers, scale invariance, and decomposability if and only if it is ordinally equivalent to (21) for some α.*[45]

4.1.3 The role of key axioms.

The result of Theorem 4 might appear at first glance to have been produced like a conjuring trick. However it is one of a number of similar results that can be generated by combinations of basic axioms listed in section 2.3.[46] The decomposability assumption induces the additive structure, and the scale invariance (or homotheticity) property induces the power-function form of the income-evaluation

[44]Note that (21) is usually defined only on $[0, \infty)$ and is undefined for zero values of x if $\alpha < 0$; negative values of x can be handled in the very special case where α is an even positive integer. For the special cases $\alpha = 0, 1$ the general form (21) becomes (20) and (19) respectively; see also Kuga (1973), Foster (1983). Kuga (1979) examines the behaviour of these measures in a simulation study, and in a further contribution (Kuga 1980) he shows that the experimental rankings of the Theil coefficient ($\alpha = 1$) are similar to those of the Gini coefficient - see subsection 4.2 below. For recent reinterpretation of generalised entropy see Chu et al. (1996), Foster and Shneyerov (1997) .

[45]See Bourguignon (1979), Cowell (1980b), Shorrocks (1980, 1984), Russell (1985), Zagier (1982).

[46]These issues of structure are discussed in Blackorby and Donaldson (1978, 1980b, 1984), Ebert (1988b). See also sections 5 and 6 below.

function in (21). A simple generalisation of the approach illustrates how crucial to the determination of the general shape of the index is the invariance assumption. Apply the scale-invariance assumption to $F^{(+k)}$;[47] then Theorem 4 will yield the modified family of indices:

$$I_{\text{int}}^{\alpha,k}(F) := \frac{1}{\alpha^2 - \alpha} \int \left[\left[\frac{x+k}{\mu(F)+k} \right]^{\alpha} - 1 \right] dF(x) \tag{23}$$

We find that as $k \to \infty$ (23) adopts the form:

$$I_{\text{K}}^{\beta}(F) := \frac{1}{\beta} \left[\int e^{\beta[x-\mu(F)]} dF(x) - 1 \right] \tag{24}$$

where $\beta > 0$ is a sensitivity parameter.[48] The family of indices (24) - usually known as *Kolm indices* (Kolm 1976a) - form the translation-invariant counterparts of the family (21) (Eichhorn and Gehrig 1982) (Toyoda 1980). The cases of (23) corresponding to $0 < k < \infty$ are usually known as *intermediate inequality* indices (Bossert and Pfingsten 1990), (Eichhorn 1988).

4.2 Distance, rank and inequality

Of course the analysis outlined in subsection 4.1 cannot be claimed as being the uniquely appropriate method of formulating an axiomatic approach to the analysis of inequality. It is also possible that considerable progress with alternative axiomatic approaches may be based upon apparently pragmatic inequality-measurement tools: indices that have an appealing intuitive interpretation usually prove susceptible to the formulation of reasonably plausible systems of axioms.

We may illustrate this point with the *Gini index* which has long played a central role in the inequality literature. This index can be expressed in a number of equivalent forms:[49]

$$I_{\text{Gini}}(F) \quad : \quad = \frac{1}{2\mu(F)} \int \int |x - x'| \, dF(x) dF(x') \tag{25}$$

[47] This is equivalent to adopting the change of variable $z := x + k$ and assuming that the inequality rankings of distributions of z are scale-invariant; alternatively one shifts the origin from which one measures income from 0 to $-k$. The role of income transformations in defining inequality concepts and their relationship with social-welfare functions is discussed further in Ebert (1996, 1997b).

[48] See the discussion in Cowell (1998).

[49] Based on Gini's mean difference - see David (1968, 1981), Gini (1921), Glasser (1961), Helmert (1876), Jasso (1979). There are other equivalent interpretations and formulae for the Gini coefficient and mean difference that are sometimes useful - see Berrebi and Silber (1984), de Finetti (1931), Dorfman (1979), Galvani (1931), Giaccardi (1950), Lerman and Yitzhaki (1984), Giorgi (1984, 1990, 1993), Stuart (1954), Yitzhaki (1982b, 1998).

$$= 1 - 2\int_0^1 L(F;q)dq \quad (26)$$

$$= \int x\kappa(x)\, dF(x), \quad (27)$$

where $x, x' \in \mathfrak{X}$ and $\forall F \in \mathfrak{F}$, $x \in \mathfrak{X}$: $\kappa(x) := [F(x^-) + F(x^+) - 1] \;/\; \mu(F)$. The Gini coefficient has a number of practical advantages: for example it deals with negative incomes (Berrebi and Silber 1985) (Chen et al. 1982) (Stich 1996) and it satisfies both the scale-invariance and translation-invariance principles.[50] Furthermore it suggests natural interpretations of income distribution and axiomatisation of inequality as may be illustrated by each of the above three forms:

- (25) presents its standard interpretation as the normalised average absolute difference between all pairs of incomes in the population. It captures the idea of "average distance" between incomes in the population according to a particular definition of distance. Replacing this definition with an alternative concept of distance will yield other inequality measures: for example the Euclidean norm will yield a measure ordinally equivalent to the variance.[51]
- (26) reveals its close link with the (relative-) Lorenz curve: the Gini is the normalised area between the curve and the 45° line in Figure 10.[52]
- (27) reveals a particularly important feature of the Gini coefficient: it is a weighted sum of all the incomes in the population where the weights $\kappa(x)$ depend on the *rank* of the income-receiving unit in the distribution $F(x)$.[53]

[50] Such scale- and translation-invariant measures are sometimes known as *compromise indices* (Blackorby and Donaldson 1980b) - see Ebert (1988a) for a general characterisation based on the L^r metric in Gehrig and Hellwig (1982) ; see also Krtscha (1994), for a related index. Examples of the axiomatic approach in the context of Gini-type indices are: Bossert (1990), Milanovic (1994), Pyatt (1976), Ranadive (1965), Takayama (1979), Thon (1982), Tendulkar (1983), Trannoy (1986).

[51] The distance concept in (25) can be seen as the counterpart of the ℓ^1 metric on $\Re^n$: $\sum_{j=1}^n |x_j - x'_j|$ for $\mathbf{x}, \mathbf{x}' \in \Re^n$. The Euclidean ℓ^2 metric is given by $\sqrt{\sum_{j=1}^n [x_j - x'_j]^2}$.

[52] Chakravarty (1988) and Shorrocks and Slottje (1995) have suggested a simple generalisation of the Gini based on this formulation; see also Mehran (1976). For other measures based on intuitive interpretations of the Lorenz curve see Alker (1970), Alker and Russet (1964), Basmann and Slottje (1987).

[53] In the case of continuous distributions $\kappa(x)$ simplifies to $\frac{2F(x)-1}{\mu(F)}$. The intuitive interpretation is as follows. Imagine you have income x: then the number of persons below and above you are proportional to $F(x)$ and $1 - F(x)$ respectively. Let $\mathcal{E}$ be the expectations operator: given that $\mathcal{E}F(x) := \int F(x)dF(x) = \int_0^1 q dq = \frac{1}{2}$, we immediately see that the form (27) is equivalent to $\frac{cov(x,F(x))}{\mathcal{E}x\mathcal{E}F(x)}$. So the Gini coefficient is the normalised covariance of income and ranks in the population (Jenkins 1988), (Lerman and Yitzhaki 1985), (Stuart 1954). It also happens that the Gini coefficient is the (normalised) Ordinary Least Squares slope of Pen's Parade.

This formulation has prompted a number of useful generalisations of the Gini coefficient - see Ben Porath and Gilboa (1994), Bossert (1988a), Donaldson and Weymark (1980, 1983), Weymark (1981), Yaari (1988), Yitzhaki (1983).[54]

5 Welfare functions

5.1 Insights from choice under uncertainty

As we noted in Section 4 the analysis of other economic problems involving probability distributions has served to inform the analysis of income distributions. So too with the topic of social welfare. The early modern literature on inequality measurement and the social evaluation of income distributions drew extensively on the parallel literature in the field of individual choice in the face of uncertainty, a cross-fertilisation which has continued.[55] The mapping from individual preferences over uncertain prospects into coherent utility functions, the formulation of riskiness and the concept of risk aversion, are all mirrored in the welfare analysis of distributional comparisons. Details of attitudes to risk are matched by "rightist", "centrist" and "leftist" interpretations (Kolm 1976a, Kolm 1976b) of attitudes to inequality associated with the structure of the contours of social-welfare functions. The resulting tools for distributional analysis are closely related to those already discussed in the context of the axiomatic approach.

5.2 Basic concepts

A social welfare function might not at first seem to be a very appealing basis for measuring income inequality. A welfare function - like utility functions in consumer theory - has an arbitrary cardinalisation; and even in the case of the special additive form (11) the scale and origin of the evaluation function u are indeterminate. However, many may be persuaded by the idea that a more equitable income distribution would be a good thing, in which case one might reasonably expect to be able to construct a link between welfare theory and inequality measurement.[56]

However a simple transformation of the SWF yields a practical tool for distributional analysis: the *equally-distributed equivalent* can be defined as a money-metric of social welfare. Use the definition of an equal distribution $H^{(\cdot)}$ - given

[54]Other applications include the application of inequality to voting mechanisms treated as co-operative games: in this context Einy and Peleg (1991) argue a role for the generalised Gini of Weymark (1981).

[55]See for example Atkinson (1970), Theil (1967), Yaari (1987, 1988).

[56]See, for example Aigner and Heins (1967), Broome (1988), Dalton (1920), Meade (1976); for a polemical case for equality as a social norm see Tawney (1964); Young (1994) sets income inequality in context with other notions of equity.

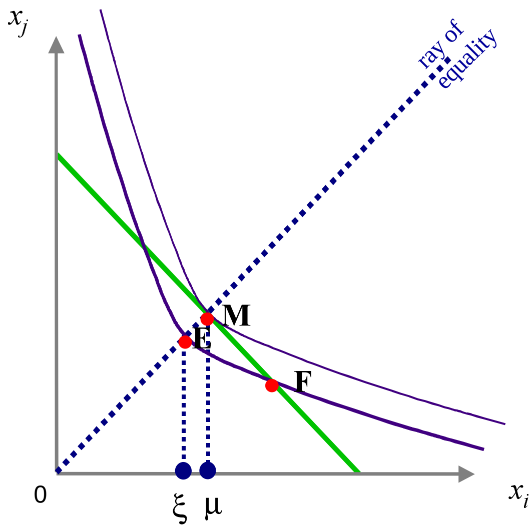

Figure 12: The equally-distributed-equivalent is less than the mean

by (8) - to provide an implicit definition of a number ξ such that

$$W(H^{(\xi)}) = W(F) \tag{28}$$

This can be used to yield the equally-distributed equivalent as a functional $\mathfrak{F} \mapsto \Re$; in other words, given a distribution F, $\xi(F)$ may be extracted from equation (28). The expression $\xi(F)$ is that income which, if it were imputed to every income-receiver in the population would yield the same level of social welfare as the actual income distribution F.[57] Figure 12 illustrates the idea. Let point F represent an income distribution in a two-person economy; then mean income μ can be found as the abscissa of the point M where the 45° line through F intersects the equality ray; the equally-distributed equivalent ξ is the abscissa of the point E where the W-contour through F intersects the equality ray. Clearly the farther along the constant-total-income line is point F from perfect equality M, the lower is ξ; the normalised gap between ξ and μ then provides a natural basis for an inequality index:

$$I_{\text{A}}(F) := 1 - \frac{\xi(F)}{\mu(F)} \tag{29}$$

The formulation (29) permits a general approach to social-welfare values interpreted as aversion to inequality: for any given income-distribution the more sharply convex to the origin is the contour in Figure 12, the greater is the gap

[57]See Atkinson (1970), Kolm (1969). The use of the equally-distributed-equivalent was anticipated by Champernowne (1952), page 610.

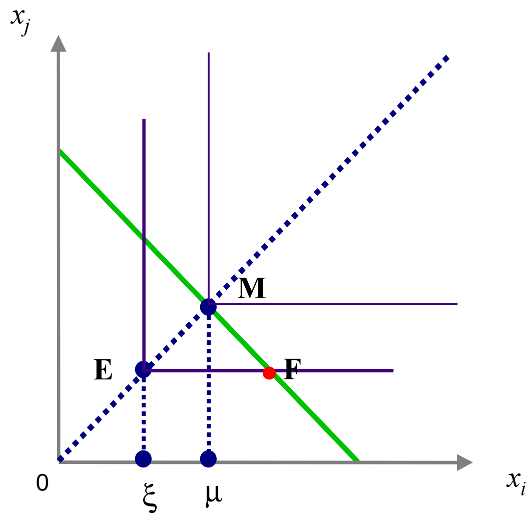

Figure 13: Extreme inequality aversion?

between ξ and μ; in an extreme case, given that welfare is assumed additively separable (11),[58] one would get situation such as Figure 13 (Hammond 1975).

However, in implementing this idea as a practical tool we need to address three specific issues:

- the derivation of an index
- the nature of inequality aversion
- the structure of the SWF

5.2.1 The Atkinson index

To obtain a specific inequality measure we need to impose more structure on W. If we also require that the principle of scale invariance hold then ξ in (29) becomes a kind of generalised mean (Atkinson 1970):[59]

$$I_{\mathrm{A}}^{\varepsilon}(F) := 1 - \frac{1}{\mu(F)} \left[\int x^{1-\varepsilon} dF(x) \right]^{\frac{1}{1-\varepsilon}} \tag{30}$$

[58] The restriction to the class (11) is important. In the absence of this, other concepts of extreme inequality aversion could be introduced - see the discussion of "super-egalitarian" criterion in Meade (1976) page 49, and the discussion of Figure 14 below.

[59] Cf. the alternative approach by Chew (1983); see also Bossert (1988b) for the case where the strict scale-invariance assumption is replaced by a more general form of invariance. The limiting form of (30) as $\varepsilon \to 1$ is $I_{\mathrm{A}}^{1}(F) := 1 - \exp\left(\int \log(x) dF(x)\right)$.

where $\varepsilon \geq 0$ is a parameter defining *(relative) inequality aversion.* A brief comparison of I_{GE}^{α} and $I_{\mathrm{A}}^{\varepsilon}$ (equations 21 and 30) shows that they are ordinally equivalent - they will have the same shape of contours in $\mathfrak{F}(\mu)$ - for cases where $\alpha = 1 - \varepsilon$.

Again it is clear that alternative sensible assumptions about structure and normalisation of W and ξ could be made which will induce alternative families of inequality measures: for example requiring that welfare comparisons satisfy translation invariance with a suitable normalisation will yield the "absolute" indices (24) instead of the "relative" measures (21) and (30).

5.2.2 Inequality aversion

The inequality-aversion concept is clearly central to the Atkinson index (30) and is implicit in the sensitivity parameters used in (21) and (24). Two issues suggest themselves: How is the aversion to inequality to be interpreted? On what is it supposed to be based?

There are at least two ways of interpreting the idea of inequality aversion - or two types of inequality aversion - which may be summarised in the questions:

1. "How should transfers from the rich to the quite-well-off be ranked against transfers from the quite-well-off to the poor?"[60]

2. "At what rate should society be prepared to trade off equality against mean income?"

Question 1 is what is captured by the sensitivity parameter in (21); question 2 is the fundamental issue of political economy highlighted by Okun (1975) and others. The two questions are, in general, not identical (Cowell 1985a) although sometimes the specification of the SWF obscures this point.

As far as the basis of inequality-aversion is concerned we could consider it to be rooted in individual distributional judgments. These could take the form of the individual valuation of an externality involving other people's incomes or living standards (Hochman and Rodgers 1969), Kolm (1964, 1969), (Thurow 1971), (Van Praag 1977) or the form of risk perceptions (Amiel and Cowell 1994a), (Harsanyi 1955). In the first case inequality aversion is determined by the marginal utility of the externality, in the latter by risk aversion;[61] in both cases social welfare can be taken as an embodiment of personal preferences, and it may be illuminating to investigate the strength of, and factors determining, inequality aversion.[62] An alternative approach is to suppose that social values, including

[60]See the discussion of the related point for inequality measures in note 3.4.3.

[61]In this case inequality measures can be interpreted as measures of riskiness of an income distribution (Dahlby 1987).

[62]See Amiel, Creedy, and Hurn (1998), Gevers et al. (1979), Glejser et al. (1977) for empirical studies on students; see also Van Praag (1977, 1978), Van Herwaarden et al. (1977), Van Batenburg and Van Praag (1980) for an ambitious research programme focusing on welfare and inequality perceptions using a specific functional form for individual welfare functions.

inequality aversion, will be revealed by public policy decisions (Christiansen and Jansen 1978), (Stern 1977), although this may run into the problems of falsely assuming coherence and rationality on the part of governments and their agents, as well as problems of specification of the SWF.[63]

5.2.3 The structure of the SWF

As we have seen, the derivation of the specific welfare-based index such as (30) required the introduction of some assumptions about the structure of W- or I-contours. However there remains another important issue of structure of the SWF which will impact upon the interpretation of inequality aversion and the relationship between inequality and social welfare.

Consider a welfare function $\hat{W}$ derived from W in the following manner

$$\hat{W}(F) = \Psi\left(\mu(F), W(F)\right) \tag{31}$$

where $\Psi : \Re^2 \mapsto \Re$ is increasing in its second argument. It is clear that $\hat{W}$ and W will have the same contours in $\mathfrak{F}(\mu)$, and therefore the same family of associated inequality measures, but also that they may have dramatically different responses to income growth:[64] the μ-dependent transformation Ψ will affect the implied trade-off between equality and mean income, and the choice of Ψ is not innocuous. To each distinct Ψ there will be a distinct value of type-2 inequality aversion, for any given type-1 inequality aversion.

What of the specific additive form of SWF (11) that we used so extensively in the distributional ranking results of Section 3? It is clear that, despite its attractive simplicity of form and the fact that it can be supported by some *a priori* ethical arguments (see footnote 26 above), it is somewhat restrictive. First, there are some sensible systems of social values W for which there is no μ-dependent transformation Ψ such that $\hat{W}$ has an additive form.[65] Second, even where it is possible to find some Ψ that permits representation of a particular SWF in the form (11), insisting on additivity of the SWF may rule out some important aspects of social values.

For example if the SWF does not belong to the restrictive class given by (11) then it is possible that W may satisfy the principle of transfers, and the principle

[63] Guerrero (1987) suggests that, given the standard Atkinson-type evaluation function ("utility function") $u(x) = \frac{x^{1-\epsilon}-1}{1-\epsilon}$ in (30) one might determine use a Box-Cox (1964) method of estimating ϵ on the assumption that "utility" is normally distributed, but it is difficult to see why this data-driven statistical procedure should be appropriate to the selection of an essentially normative parameter.

[64] Consider for example the case where W is additive with scale-invariant contours and Ψ is simply the transformation by which one extracts $\xi(F)$ from $W(F)$ in (28), normalised by the mean: $\hat{W}(F) = \Psi\left(\mu(F), W(F)\right) = \xi(F)/\mu(F)$. W will increase with proportional increases in all incomes; $\hat{W}$ will not.

[65] See Newbery (1970). The additive separability assumption rules out "positionalist" welfare functions (Fine and Fine 1974a, 1974b) (Gärdenfors 1973).

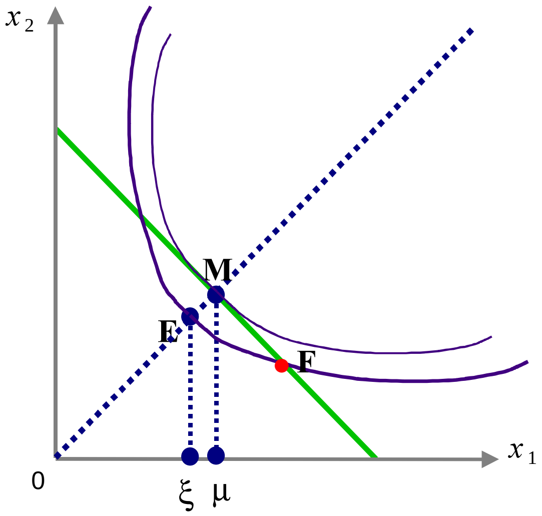

Figure 14: Inequality aversion with a non-monotonic W

of uniform income growth, but violate the monotonicity principle. An example of this situation is given in Figure 14. The distribution represented by point F is the same as in Figure 12. By construction the equally-distributed-equivalent income - and hence inequality - is also the same as in Figure 12; but it is evident that continually increasing one individual's income has a dramatically different impact in the two cases: in Figure 14 this raises social welfare if the individual has only a modest income, but may reduce welfare if the individual is already rich.

5.3 Social welfare and inequality

The SWF opens up an "indirect" approach to inequality. The mapping $W \rightarrow I$ presupposes that social values on distributional questions have already been settled;[66] and thus the "inequality map" is predetermined by the contours of the SWF. However the formal welfare-inequality link can be exploited in a number of other ways.

[66]On the social-welfare basis for redistribution see for example Kolm (1995, 1997b), Roemer (1996), Zajac (1996).

5.3.1 Reduced-form social welfare

Introduce the *reduced-form version* Ω of the SWF[67] implicitly defined by

$$W(F) = \Omega\left(\mu(F), I(F)\right) \tag{32}$$

where $\Omega : \mathfrak{X} \times \Re \mapsto \Re$ is increasing in its first argument and decreasing in its second argument; Ω encapsulates the concept of an equality-total-income trade-off (Dutta and Esteban 1992). The form (32) suggests some ways forward in examining the link between welfare and inequality, and some difficulties in the relationship.

For a start the relationship (32) immediately opens the way for the discussion in welfare terms of *ad hoc* approaches to inequality measurement (Aigner and Heins 1967), (Bentzel 1970), (Champernowne 1974), (Kondor 1975).[68] Even though a particular index may have been constructed for reasons of statistical convenience, mathematical elegance or seat-of-the-pants intuition, it may yet have interesting welfare properties that commend its use in a variety of applied welfare-economics problems.

Furthermore from the relationship (32) and the analysis in section 4 it suggests that we might construct an approach to inequality in the "reverse direction". This $I \to W$ mapping is particularly useful when one has a clear idea on the basis for an inequality index and wants welfare rankings to be consistent with this inequality criterion, but otherwise is unable to specify a SWF completely - see 5.3.3 below.

Now for one of the principal difficulties. For a given Ω let $W^* := \Omega\left(\mu, I^*\right)$ denote the social-welfare functional corresponding to a given inequality functional I^*: the ordinal equivalence of I and I^* does not entail ordinal equivalence of W and W^* (Blackorby and Donaldson 1984), (Ebert 1987).

5.3.2 The cardinalisation issue

It might seem that the ordinal properties of inequality measures alone contain the essentials of the problem: for example, given the ordinal equivalence of I_{GE}^{α} and $I_{\mathrm{A}}^{\varepsilon}$ for $\alpha = 1 - \varepsilon$ we can mechanically transform one index into the other for any given distribution F using the formula

$$I_{\mathrm{GE}}^{\alpha}(F) = \frac{\left[1 - I_{\mathrm{A}}^{\varepsilon}(F)\right]^{\alpha} - 1}{\alpha\left[\alpha - 1\right]} \tag{33}$$

Need anything more be said? This oversimplification is misleading in two respects. First, cardinalisation has an important role to play in decomposition

[67] This is the term used by Champernowne and Cowell (1997). See also the term "abbreviated social-welfare function" used in Lambert (1993), Chapter 5. See also Blackorby, Donaldson, and Auersperg (1981) who discuss the conditions under W which is expressible in this form.

[68] For other approaches exploiting the connection between social welfare and inequality see Dagum (1990, 1993).

analysis - see section 6 below; second, as we have just seen in 5.3.1, there are problems regarding the welfare interpretation of inequality measures.

Take the issue of interpreting inequality measures and consider the question: what constitutes an "important" change in inequality? The question is implicitly raised in, for example, comparative studies of the development of the income distribution over different time periods using an inequality index as a performance indicator. Here there appears to be a rôle for the social welfare function. For example we could try using the welfare function to get an income-equivalent of a particular change in measured inequality using (29).[69] However, not only is a benchmark for numerical comparisons required ("is a 1 percent increase in the index 'big'?") but also a criterion for comparing the magnitude of one distributional change with another; the validity of a statement such as "inequality I increased more in period t_1 than it did in period t_2" is dependent on a particular cardinalisation of I;[70] but in order to interpret this in welfare-terms we need to make a non-trivial assumption about the structure of social welfare. To see this use the definition of ordinal equivalence for inequality measures (18) and the reduced-form social welfare function (32) to get the general relation:

$$W(F) = \Omega\left(\mu(F), \psi\left(\mu(F), I(F)\right)\right) \tag{34}$$

from which we obtain:

$$\frac{d\mu(F)}{dI(F)} = -\frac{\psi_I}{\frac{\Omega_\mu}{\Omega_I} + \psi_\mu} \tag{35}$$

as the change in average income that exactly offsets a given inequality change. This income-equivalent depends on two factors:

- the inequality cardinalisation ψ,
- the shape of the reduced-form SWF Ω.

The first of these may be considered to be fairly arbitrary, and we will see some pragmatic arguments for particular "natural" inequality cardinalisations below; the second is not just arbitrary, but represents some basic social issues: the assumption of the additivity of W, or of the monotonicity of W, plays a fundamental role in the evaluation of inequality changes. Disentangling the two factors is inevitably problematic, and these issues impinge upon the topics considered next in 5.3.3 and 5.3.4.

[69] For a given change in I_{A} the method requires finding the offsetting change in μ that leaves $\xi(F) = \mu(F)\left[1 - I_{\mathrm{A}}(F)\right]$ unchanged - see Cowell (1995) page 132.

[70] To see this, consider I and $\hat{I}$ such that $\hat{I} = I^2$ and three distributions F_0, F_1, F_2 such that $I(F_0) = 0.8, I(F_1) = 1, I(F_2) = 1.19$: measure I indicates that $F_0 \to F_1$ is a greater change in inequality than $F_1 \to F_2$; measure $\hat{I}$ indicates the opposite. Note that, although it does not invoke the use of a SWF, the procedure suggested by Blackburn (1989) also depends on the cardinalisation of inequality.

5.3.3 From inequality to welfare

The problem of building a full welfare function, or class of welfare functions, on the basis of a pre-specified inequality index requires additional information to fill an important gap. The principal point is that the conventionally defined ordinal inequality measure is defined on $\mathfrak{F}(\mu)$ and so, of itself, does not encode any information about what happens as one income increases. In particular we might wonder whether the resultant welfare function(s) satisfy monotonicity. In the *separable* case consisting of measures that are ordinally equivalent (in the sense of 18) to the form:

$$\int \phi(x)dF(x) \tag{36}$$

the issue is fairly transparent, if Ω and ϕ are differentiable. If the inequality measure I is cardinally equivalent to (36) then all we need to do is to ensure that a fairly mild condition on the slope of the reduced form welfare function is satisfied:

$$-\frac{\Omega_\mu}{\Omega_I} > \max \phi_x(x) \tag{37}$$

(where the subscripts denote partial derivatives); measures that ordinally, but not cardinally, equivalent to (36) require a slightly modified form of (37). The nonseparable case is a little more difficult, since it includes indices such as the Gini coefficient which is non differentiable.[71] However Amiel and Cowell (1997) demonstrate that a version of (37) applies as a bounding condition in this case too.

5.3.4 Inequality and growth

What happens to inequality as incomes grow? Apart from the practical question of whether the historical process of economic growth is typically accompanied by increasing income disparity (Kuznets 1955) there is also an issue of interpretation: given a specific hypothetical change in one or more persons' income what would we "reasonably" expect to happen to inequality? (Fields 1987), (Glewwe 1990), (Kolm 1976b).

The problem can be interpreted in standard individualistic welfare terms, as illustrated in Figure 15: Suppose the income distribution currently is given by point A. In what direction from A should an increase in Irene's and Janet's incomes be made in order to keep inequality unchanged? Call this the *transformation direction.* Two obvious suggestions for the transformation direction would be $\overrightarrow{\mathrm{AB}}$

[71]Examples of reduced-form Gini-SWFs in the literature include $n^2[1-I]\mu$ (Sheshinski 1972), $\log \mu - I$ (Katz 1972), and $\frac{\mu}{1+I}$ (Kakwani 1986) all of which satisfy monotonicity, and $\frac{1-I}{1+I}\mu$ (Chipman 1974, Dagum 1990) which does not.

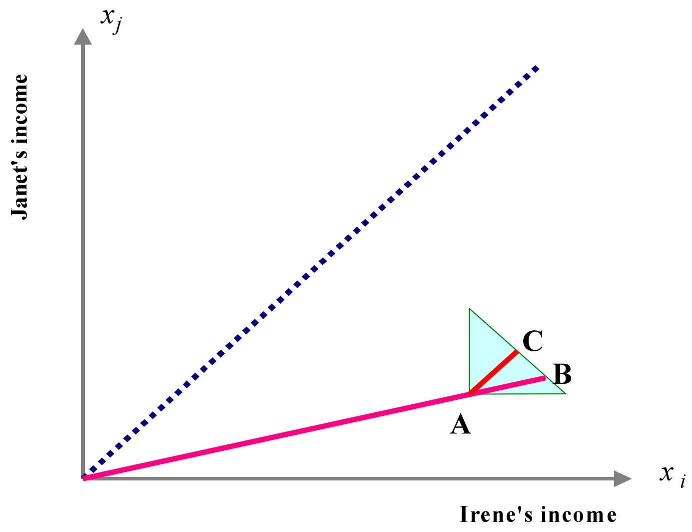

Figure 15: Which direction leaves inequality unchanged?

(an equi-proportional increase in both persons' incomes) and $\overrightarrow{\mathrm{AC}}$ (an equal absolute increase in both persons' incomes): but any direction in the shaded triangle would be valid. If there is a uniform transformation direction for all income distributions then this will induce a specific structure on the inequality measure: if inequality remains everywhere unchanged under scale transformations (like $\overrightarrow{\mathrm{AB}}$ in the above example) this will force the inequality measure to be *scale-independent*, $I\left(F^{(\times k)}\right) = I(F)$; if inequality remains everywhere unchanged under translations (like $\overrightarrow{\mathrm{AC}}$ in the above example) this produces a *translation-independent* inequality measure, $I\left(F^{(+k)}\right) = I(F)$.[72] More generally we could imagine a general iso-inequality map where the transformation direction changed at different parts of the set of possible income distributions, such as the two examples in Figure 16: case (a) depicts a situation where, at low incomes, equal absolute additions *increase* inequality, at moderate incomes there is (local) translation-independence, and at high incomes, equal proportional additions *reduce* inequality; case (b) depicts the situation suggested by Dalton (1920) who argued that both absolute and proportionate additions to income would reduce inequality.[73]

Alternatively the relationship between inequality and income growth (or decline) may be taken as fundamental to the definition of the *meaning* of inequality: Temkin (1986, 1993) has argued that inequality should be formulated in terms of

[72] In general a uniform transformation direction will yield a form ordinally equivalent to (23) for $k \geq 0$. Note that the properties of scale- and translation-independence are stronger than those of scale- and translation-*invariance* introduced in 2.3 above.

[73] Questionnaire-experimental evidence suggests support for the Dalton view (Amiel and Cowell 1999a).

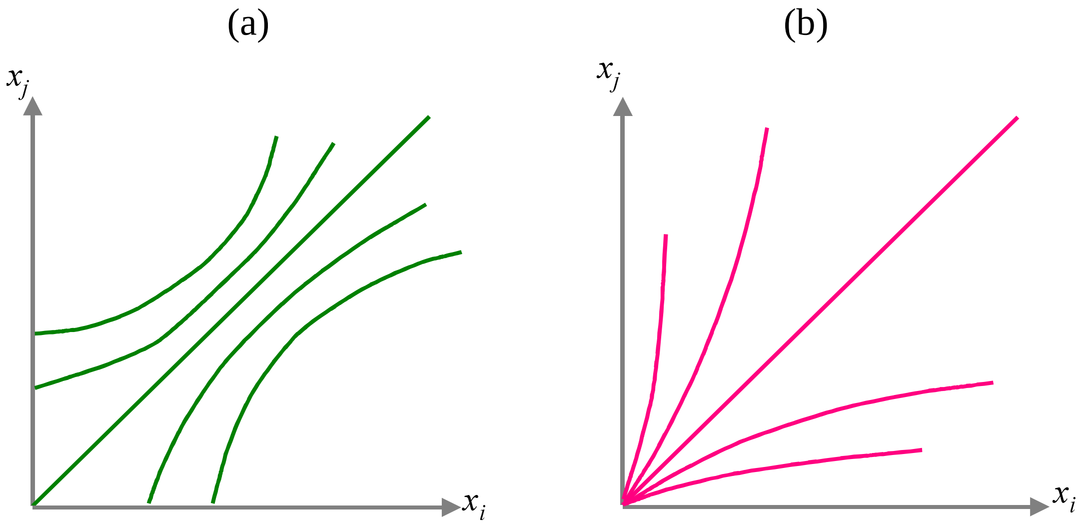

Figure 16: Inequality maps: (a) a variety of transformation directions; (b) the Dalton conjecture

"complaints", which Temkin further rationalises in terms of the changing pattern of income differences as individuals migrate between low-income and high-income groups.[74]

5.3.5 Relative deprivation

Relative deprivation is a sociological concept whose social-welfare analytic counterpart may be seen as having grown out of the relationship between inequality measures and SWFs; it has some structural similarity to the formal work on poverty measurement - see 6.3.2 below.

In a sense the economic insights on the topic of relative deprivation have made a virtue of the necessity of focusing upon the Gini coefficient: the very features of the Gini that make it awkward for some branches of the modern literature on inequality (see, for example, section 6 below) make it particularly attractive for embodying the relative deprivation concept of Runciman (1966).[75] Like the Temkin concept of complaint discussed in 5.3.4 relative deprivation seems to lend itself to a natural expression in terms of income differences: the rôle of rank in defining the Gini coefficient can surely be reinterpreted in terms of social disadvantage.

Suppose the relative deprivation experienced by a person with income x' is

[74] See also the "isolation" and "elitism" concepts in Fields (1993), Figini (1996) and the discussion in Fields (1998).

[75] However, for an alternative view see Podder (1996).

measured by

$$\int_{x'}^{\infty} [x - x'] \; dF(x) \tag{38}$$

then the aggregated value of this over the distribution F is

$$\mu(F) - 2\int_0^1 C(F;q)dq \tag{39}$$

which is simply $\mu(F)I_{\text{Gini}}(F)$ - the "absolute Gini".[76]

The form (39) shows the close relationship between this interpretation of deprivation and GLC rankings (Hey and Lambert 1980) and a number of straightforward generalisations of the concept have been proposed - see Berrebi and Silber (1985, 1989), Chakravarty (1998), Chakravarty and Chakraborty (1984), Chakravarty and Mukherjee (1997), Hey and Lambert (1980), Stark and Yitzhaki (1988), Yitzhaki (1979, 1980, 1982a).

6 The structure of inequality

6.1 The basic problem

The discussion of the basic axioms of distributional analysis included decomposability as one of the fundamental properties that might be considered in the formal approach to income distribution. However, beyond its use in the specification of some convenient inequality tools and the discussion of the additivity of the SWFs, the issue of inequality decomposition raises a number of questions concerning the structure of distributional comparisons. These resolve into two major types of problem:

- *By population subgroup.* We assume that individuals may be distinguished by personal or group attributes which serve to partition the population into distinct sub-populations. This can be useful in the analysis of the relationship between inequality in a whole country and inequality within and between its regions, or between inequality in a heterogeneous group of persons and inequality within and between subgroups categorised by gender,

[76] Individual deprivation (38) may be written

$$\mu(F) - C(F;F(x)) - x + xF(x).$$

integrating this over the distribution F we get

$$= -\int_0^1 C(F;q)dq + \int_0^\infty x \int_0^x dF(y)dF(x);$$

a rearrangement of variables then gives (39).

ethnicity and the like. It is almost essential to attempts to "account for" the level of, or trend in, inequality by components of the population.

- *By income source.* For example one might we wish to relate the inequality of total income to the inequality of income from work, inequality of income from property and so on.

In other words the two types of decomposition involve looking at the structure of inequality by components of the population and by components of income. To implement either of these approaches one needs to recognise the multidimensional character of the underlying problem of distributional comparison, which we will be discussing further in Section 7; in particular decomposition by income source is examined in subsection 7.2.[77]

6.2 Approaches to decomposition by subgroup

Let a partition consist of a collection of a finite number J subgroups

$$\Pi = \{N_1, N_2, ..., N_J\}, \tag{40}$$

such that a proportion p_j of the population belong to subgroup j, $j = 1, 2, ..., J$; let $F^{(j)}$ be the income distribution in group j and $s_j := \frac{p_j \mu(F^{(j)})}{\mu(F)}$ be the income share of group j. Three issues need to be clarified:

- the exact requirements of decomposability,
- the type of partitions that are admissible,
- the nature of "between-group" inequality.

6.2.1 Types of decomposability

The definition of decomposability that we have used so far (see page 12) can be expressed in a number of equivalent forms. One of these is the *subgroup consistency* property which requires that inequality overall $I(F)$ can be rewritten in terms of any partition using the basic decomposition relation:

$$I(F) = \Phi\left(I\left(F^{(1)}\right), I\left(F^{(2)}\right), ...I\left(F^{(J)}\right); p_{1,}p_2, ..., p_J; s_{1,}s_2, ..., s_J\right) \tag{41}$$

where Φ is increasing in each of its first J arguments (Shorrocks 1984, 1988). This can be seen as a minimal requirement for decomposability by subgroup: without this property one could have the remarkable situation in which inequality in every subgroup rises (while mean income and the population shares remain unchanged)

[77]For a recent discussion and overview of the issues see Deutsch and Silber (1999), Morduch and Sicular (1996).

and yet overall inequality *falls*. However, one might wish for a more demanding interpretation of decomposability, and so let us consider two ways of strengthening the subgroup consistency requirement.

Additive decomposability requires:

$$I(F) = \underbrace{\sum_{j=1}^{J} \omega_j I\left(F^{(j)}\right)}_{\text{[within group]}} + \underbrace{I\left(F_{\Pi}\right)}_{\text{[between group]}}, \tag{42}$$

and

$$\omega_j = w(p_j, s_j) \geq 0, \tag{43}$$

where the distribution F_{Π} will be discussed in 6.2.3 below.

One might perhaps require a yet more demanding interpretation of decomposability by adding to (42) the additional restriction

$$\sum_{j=1}^{J} \omega_j = 1 \tag{44}$$

which is perhaps an "accountant's approach" to decomposition: the weights in the within-group component sum exactly to 100 percent.

As we have seen in Section 4 (page 24) if the basic consistency requirement is imposed for any arbitrary partition this will ensure that the measure must take a form that is ordinally equivalent (in the sense of 18) to the form (36); so if one also requires the property of scale invariance one obtains the GE class (21) and the weights in (42) take the form

$$\omega_j = \omega(p_j, s_j) = p_j^{1-\alpha} s_j^{\alpha}. \tag{45}$$

If one further requires the property (44) then only two measures are available: the MLD index (20) where the weights are population shares ($\alpha = 0$ in 45), and the Theil index (19) where the weights are income shares ($\alpha = 1$). (Berry et al. 1981, 1983), (Bourguignon 1979), (Cowell 1980b), (Shorrocks 1980), (Theil 1979b, 1979a), (Yoshida 1977).

6.2.2 Types of partition

In 6.2.1 we implicitly assumed that every sort of attribute partition Π of the population was valid. In some applications of decomposability it may be appropriate to consider a more restrictive subclass of partitions. In particular consider the concept of a *non-overlapping* partition in which all the constituent subgroups can be strictly ordered by their members' incomes; Figure 17 illustrates this for the case $J = 2$: in case (a) every member of subgroup N_1 has an income less than any member of N_2; in case (b) N_1 "overlaps" N_2 in terms of income ranges (Ebert 1988c).

In the light of this distinction consider the problem of decomposing an inequality index that is excluded from the cases considered under 6.2.1 above: the

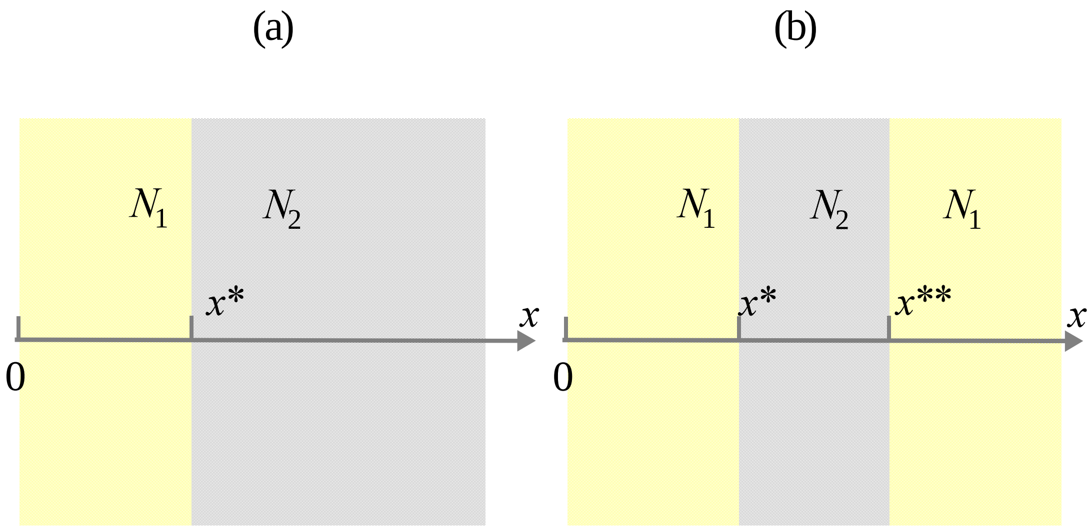

Figure 17: (a) Non-overlapping partition; (b) overlapping partition

case of the Gini index. It is well known that in general the Gini is not decomposable in the sense of subgroup consistency; if we attempt an exercise similar to that of equation (42); instead of a neat breakdown into two components we find three terms: a within-group component, a between-group component and an interaction term.[78] Whether the presence of this interaction term means that the Gini coefficient is "decomposable" in some more general sense is a moot point.

What is particularly interesting to see is why, and under what circumstances, the Gini coefficient potentially gives rise to problems. Imagine that there is a small mean-preserving change in the distribution, let us say a transfer from a person with income x to someone with income x'. Inspection of the form (27) of the Gini coefficient reveals that the effect of this depends on the expression

$$\kappa(x') - \kappa(x) = \frac{2}{\mu(F)}\left[F(x') - F(x)\right] \tag{46}$$

for continuous distributions. Contrast this to the corresponding impact of such a distributional change upon a measure that is ordinally equivalent to an additively separable index (equations 18 and 36): it would be proportional to

$$\phi_x(x') - \phi_x(x). \tag{47}$$

[78] See for example Anand (1983) (appendix), Bhattacharya and Mahalanobis (1967), Fei et al. (1979), Mangahas (1975), Mehran (1975), Piesch (1975), Pyatt (1976), Rao (1969), Silber (1989). For recent discussion on attempts to decompose the Gini coefficient see Dagum (1997) and the discussion by (Deutsch and Silber 1997) of Gini's concept of "trasvariazione" and the relationship to distance between income distributions.

Observe that (47) only needs minimal information about the specific affected income values x and x' ; but for the Gini we find that (46) requires more detailed information about the distributions to which the affected persons belong: the effect of the transfer depends on the *rank* of the affected individuals in the relevant distributions. It is clear that the term $F(x') - F(x)$ will have the same value as $F^{(j)}(x') - F^{(j)}(x)$ if the relevant partition is non-overlapping (left-hand side of Figure 17), but that the two values may differ if the partition is overlapping. For example consider x and x' in Figure 17(b) such that $x < x^*$ and $x^{**} < x'$: clearly $F(x') - F(x) > F^{(1)}(x') - F^{(1)}(x)$. Now imagine a more complex mean-preserving change within $F^{(1)}$: the impact on the within-group Gini will depend on some aggregate of a collection of pairwise transfers like $F^{(1)}(x') - F^{(1)}(x)$ and the impact on the overall Gini will depend on the aggregate of the corresponding collection of pairwise transfers $F(x') - F(x)$; there is no guarantee that these two these two aggregates will have the same sign, so that the Gini of $F^{(1)}$ might decrease while the Gini of F increased. In sum the Gini coefficient decomposes in the sense of (41) and (42) only if Π is nonoverlapping, in which case the interaction term mentioned above will vanish.

6.2.3 Between-group inequality

What is the meaning of the between-group inequality component denoted by the distribution function F_Π in (42)? Perhaps the obvious answer is to suppose that the between-group distribution is a step function

$$F_\Pi(x) = \sum_{\ell=1}^{j} p_\ell \text{ if } x \geq \mu\left(F^{(j)}\right) \tag{48}$$

where $\mu\left(F^{(1)}\right) \leq \mu\left(F^{(2)}\right) ... \leq \mu\left(F^{(J)}\right)$.[79] This is equivalent to assuming that all the probability mass in group N_j is concentrated at the mean $\mu\left(F^{(j)}\right)$. However, there are other possibilities. If the decomposable inequality index is explicitly based upon a social welfare function - such as (29) - then Blackorby et al. (1981) suggest that the appropriate representative income for each subgroup is its equally-distributed-equivalent income $\xi\left(F^{(j)}\right)$ rather than the mean.[80] This decomposition scheme can be expressed by replacing the functional μ by ξ throughout (48) - see also Foster and Shneyerov (1997).

6.2.4 The importance of decomposability

Decomposability of inequality might appear to be a luxury item, additional to other more basic criteria for selecting an inequality measurement tool. The issue of whether it is worth affording this luxury resolves into two questions:

[79] Cf. equation (7).
[80] See also Ebert (1997a).

- *Does decomposability matter?* Some commonly-used inequality measures do not satisfy even the minimal consistency properties such as the relative mean deviation[81]

$$I_{\mathrm{RMD}}(F) := \int \left| \frac{x}{\mu(F)} - 1 \right| dF(x) \tag{49}$$

and the logarithmic variance

$$I_{\mathrm{logvar}}(F) := \int \left[\log \left(\frac{x}{\mu(F)} \right) \right]^2 dF(x). \tag{50}$$

If these indices are used in empirical studies of inequality-decomposition it is difficult to avoid the conclusion that the wrong tool is being used for the job.

- *Does a particular decomposition matter?* If one is concerned merely with the ordinal properties of inequality measures then the subgroup consistency requirement (41) may be all that is required. However (42) suggests a "natural" cardinalisation for decomposable measures, but the ordinal-equivalence function ψ in (18) can be used to derive to derive decomposition formulae for alternative cardinalisations (Das and Parikh 1981, 1982). As we have seen, more than one logical way of defining the between-group components is available for a given partition, but it is important that the precise assignment of weights and the components in the decomposition are assigned in a fashion that is consistent under alternative partitions: for example, where one wants to carry out multilevel decompositions (say by age *and* gender *and* region...), the within-group/between group definition in the finest partition should be consistent with that used in other, coarser partitions (age-and-region, or age alone perhaps) (Adelman and Levy 1984) (Cowell 1985c). The type of decomposition that is appropriate - the cardinalisation, the partition, the definition of between-group inequality will ultimately depend on the economic question which one is trying to answer.

6.3 Applications

6.3.1 "Explaining" income inequality

Consider the problem of "accounting for" or "explaining" alluded to on page 37. It seems intuitively reasonable that some specific partitions are more "important" than others in the analysis of a particular economy's income distribution.

[81] See for example Cowell (1988a). The same difficulty affects other similar measures such as those suggested by Eltetö and Frigyes (1968) (see also Addo 1976, Schutz 1951), and the variance of logarithms found by replacing the term $\mu(F)$ in (50) by the geometric mean. Moreover the logarithmic variance and the variance of logarithms do not satisfy the principle of transfers everywhere (Cowell 1995), (Creedy 1977), (Ok and Foster 1997).

There are a number of ways of quantifying this (Cowell 1984) (Jenkins 1995), and the framework of analysis in section 6.2 should provide some help. For any distribution F and any partition Π consider the index

$$R(F,\Pi) := 1 - \frac{I_{\text{WITHIN}}(F,\Pi)}{I(F,\Pi)} \tag{51}$$

where I_{WITHIN} is the within-group inequality component for a particular cardinalisation of inequality and a given definition of between-group inequality: in the case of a measure expressed in additively separable form this is the first term on the right-hand side of (42). Given two personal or social attributes a and b by which one might - separately or jointly - partition the population we obviously have

$$\left.\begin{array}{l} R(F,\Pi_{a\&b}) \geq R(F,\Pi_a) \\ R(F,\Pi_{a\&b}) \geq R(F,\Pi_b) \end{array}\right\}. \tag{52}$$

where, for example, $\Pi_{a\&b}$ refers to the fine partition by both attribute categories. Using the Atkinson-type inequality index (30) for a variety of values of inequality aversion, Cowell and Jenkins (1995) show the impact on the R index of alternative assumptions about cardinalisation and between-group inequality, and that the amount of inequality "explained" by characteristics such as age, ethnicity and gender is relatively modest.

6.3.2 Poverty

As we have seen in the above discussion there are a number of connections between the modern theory of inequality measurement and poverty analysis. (Cowell 1988b), (Le Breton 1994), (Osmani 1982), (Sen 1976). One of the principal threads connecting the two is the structural analysis of the type considered in Section 6.

An operational approach to poverty requires the specification of a poverty line x^* : this may be an unique exogenously given value, some functional of the distribution F, or a set of possible values. Given x^* there is a fundamental partition of the population into poor and non-poor - a special case of the nonoverlapping partition discussed in 6.2.2. Now, in the case of the inequality applications that we have considered thus far, the anonymity axiom induces a symmetry of treatment of the component subgroups. However, in the case of the fundamental poor/non-poor partition this may be inappropriate: the nature of the poverty problem is such that one specifically wants to treat the members of the two groups differently. For this reason the *focus axiom* is introduced:[82] a perturbation of F that affects only the incomes of the non-poor should leave the poverty index unaltered.

[82] Notice that the anonymity axiom remains valid.

Based on this the standard approach is to construct an ordering of distributions of poverty gaps $g := \max\{0, x^* - x\}$, a device which effectively filters out the (irrelevant) information about the non-poor (Danziger and Jäntti 1998), (Jenkins and Lambert 1997), (Shorrocks 1998). The distribution of poverty gaps F^* is a simple transform of F, censored at the poverty line (Takayama 1979), and many of the tools that are commonly applied to income distribution may be adapted to the problem of poverty measurement. Distributional dominance as discussed in Section 3 translate into criteria for poverty dominance (Atkinson 1987) (Foster and Shorrocks 1988a, 1988b) and standard families of non-overlapping-decomposable inequality indices translate into poverty indices (Foster 1984), (Blackorby and Donaldson 1980a), (Clark, Hemming, and Ulph 1981), (Foster et al. 1984) (Sen 1976). As an example of the latter consider the Foster et al. (1984) indices given by

$$\int \left[\frac{g}{x^*}\right]^a dF^*(g) \tag{53}$$

where $a \geq 1$ is a sensitivity parameter:[83] the family resemblance between (53) and the inequality indices (21) and (30) is evident.

7 Multidimensional approaches

7.1 The General Problem

As we have briefly noted in discussing the main body of analysis on inequality measurement, there is a good case for considering the problem of analysing income distributions as essentially one of multivariate rather than univariate analysis. That being the case we ought to consider as our fundamental tool a distribution function F - a typical member of $\mathfrak{F}_r$, the set of r-dimensional distributions; F is the joint distribution of variables $x_1, x_2, ..., x_r$. Let F_j be the marginal distribution of x_j and $\bar{F}$ the distribution of $\sum_{j=1}^{r} x_j$. Some aspects of the multivariate problem have already been developed in sections 2.1 and 6.2 dealing with particular issues in the way households or families are to be distinguished by characteristics other than income; the remaining issues lie in three broadly defined areas:

1. The extension of ranking principles and measures. (Bradburd and Ross 1988), (Flückiger and Silber 1994), (Maasoumi 1986, 1989, 1999), (Rietveld 1990), (Tsui 1995).

2. Questions involved in multidimensional aggregation of income components (Maasoumi and Nickelsburg 1988).

[83]The restriction $a \geq 1$ is required to ensure that (53) does not violate the principle of transfers: i.e. that a transfer from a poor person to someone less poor could not reduce measured poverty. However the headcount ratio (which violates the principle) can be obtained as the special case of (53) where $a = 0$.

3. The applications of multidimensional analysis to general welfare criteria and to specific welfare-economic issues. (Kolm 1973, 1977), (Foster, Majumdar, and Mitra 1990).

The general problem (1 above) is inherently complex, principally because one has to take into account the interaction amongst variates, whether interpreted as the interrelations between income and non-income personal attributes, or as multiple components of income involved in a multidimensional generalisation of the Lorenz curve and related concepts (Atkinson and Bourguignon 1982), (Koshevoy 1995). However, progress with interpretable results is possible in a number of interesting special cases. We will examine first the issue arising under item 2, and then two aspects of welfare economic issues (item 3).

7.2 Decomposition by income source

Assume that income x consists of two components; then, taking a bivariate distribution $F \in \mathfrak{F}_2$, we have by definition an elementary variance decomposition:

$$\mathrm{var}(x_1 + x_2) = \mathrm{var}(x_1) + \mathrm{var}(x_2) + 2\mathrm{cov}(x_1, x_2). \tag{54}$$

Using (54) we find that the standard inequality measure (22) can be written as

$$I_{\mathrm{CV}}(\bar{F})^2 = \lambda_1^2 I_{\mathrm{CV}}(F_1)^2 + \lambda_2^2 I_{\mathrm{CV}}(F_2)^2 + \; 2\lambda_1\lambda_2 I_{\mathrm{CV}}(F_1) I_{\mathrm{CV}}(F_2)\; \rho(F) \tag{55}$$

where $\lambda_j := \frac{\mu(F_j)}{\mu(F)}$ measures the "importance" in income terms of income type j, and $\rho(F)$ is the correlation coefficient for the bivariate distribution F. The technique can be extended with some elaboration to cases with $J > 2$ income components.

Of course it is not to be expected that other arbitrary inequality measures will have such a neat exact formula for decomposition by income source. However it is interesting to see whether it is possible to assign a decomposition rule to determine the impact of the inequality of income component j upon the inequality of total income. There are two problems here. First, even in cases which appear to permit this sort of decomposition (typically those that can be written as a linear function of income) the result can be messy. For example there is considerable interest in applying the technique to the Gini coefficient (Podder 1993, 1995) (Sandström 1983) (Silber 1989) (Fei et al. 1978) (Lerman and Yitzhaki 1985) (Pyatt et al. 1980) (Stark et al. 1986, 1988); using (27) we get

$$I_{\mathrm{Gini}}(F) = \int x\kappa(x)\; d\bar{F}(x) = \sum_{j=1}^{J} \left[\int x_j \kappa(x)\; d\bar{F}(x) \right]. \tag{56}$$

The term inside the brackets in (56) is typically used as the basis for specifying the "contribution" to inequality of income component j; but this term is not a true inequality index.[84] Second, without further restriction on the decomposition rule, the assignment of these inequality-contributions is non-unique (Shorrocks 1982), (Chakravarty 1990).

7.3 Income and Needs

Up until now we have assumed that the issue of differing needs could be handled by a transformation of the income variable. This is not entirely satisfactory because equivalence scales with different parameters or different methods of equivalisation could lead to dramatically different conclusions on welfare comparisons and because there is no generally accepted method of deriving a unique equivalence scale (Coulter et al. 1992b, 1994a, 1994b), Jenkins and Cowell (1993, 1994). An alternative approach would be to see how much can be said about distributional comparisons without precommitment to a particular equivalence scale.

Let us make use of the (attributes-income) method of describing individuals that we introduced in subsection 2.1. Instead of assuming the existence of an equivalising function χ suppose instead that the population can be unambiguously partitioned into J different needs categories. Category j is a set N_j. Then a simple extension of the additive form of the SWF (11) yields:

$$W(F) = \sum\nolimits_j p_j \int_{\mathbf{a}\in N_j} u(y) dF(\mathbf{a}, y) \tag{57}$$

where p_j is the proportion of the population that are of type j. This is no more than a relabelling. If we allow for the possibility that one has a "categorical" social evaluation function - the income evaluation u also depends upon each person's needs category - we then would have:[85]

$$W(F) = \sum\nolimits_j p_j \int_{\mathbf{a}\in N_j} u(j, y) dF(\mathbf{a}, y) \tag{58}$$

Assume that it is possible to label the needs categories j unambiguously and arrange them in descending order of need, independently of income. This requires that the marginal social-utility gap between needs levels become smaller at high levels of incomes; so for every category j the expression

$$\frac{\partial u(j, y)}{\partial y} - \frac{\partial u(j+1, y)}{\partial y} \tag{59}$$

[84] This is usually known as a concentration coefficient - see Lambert (1993), page 50.

[85] Cf. the "fundamental utility" in Kolm (1971, 1994, 1997a). If social welfare were expressed in Harsanyi-type terms (see footnote 14 above) then we can interpret (58) as:

$$W(F) = \sum\nolimits_j \Pr\{\mathbf{a} \in N_j\} \mathcal{E}\{u(j, y) | \mathbf{a} \in N_j\}.$$

The anonymity axiom applies within each needs category N_j but not between them - Cf. the "partial symmetry" concept in Cowell (1980b) .

should be positive and decreasing in y.

Let us denote by $\mathfrak{W}_3$ the subclass of $\mathfrak{W}_2$ such that this condition on (59) holds. Then we can introduce the concept of *sequential generalised-Lorenz dominance.* Let $F^{(\leq j)}$ denote the distribution covering the subpopulation of the first j most needy groups. Notice the contrast here with subgroup- decomposition analysis: in (41) each observation in the income observation in the whole population appears in one and only one subgroup distribution; here if the attributes of a particular person i belong to N_j then his income will appear in the distribution $F^{(\leq j)}$ and also in $F^{(\leq j+1)}$, $F^{(\leq j+2)}$,... .

Then we have:[86]

Theorem 5 $W(G) \geq W(F)\ \forall (W \in \mathfrak{W}_3)$ *if, and only if*

$$G^{(\leq j)} \succeq_C F^{(\leq j)}\ \forall j = 1, 2, ...$$

Theorem 5 neatly extends the second–order-dominance criterion to the heterogeneous household case, although it is somewhat demanding since, apart from the stringent needs-ranking condition (59), it also requires that the proportion of households in each of the needs categories is the same in the two distributions under comparison: this restriction has been relaxed in Jenkins and Lambert (1993).

7.4 Distributional change

The multidimensional approach to income distribution permits us to address a class of "from-to" questions that are a natural extension of the problem of inequality measurement. Perhaps the most obvious example of this class of question is the case highlighted on page 9: there may be substantial change in the distribution of income from period 1 to period 2, even though the two marginal distributions are identical; we may want to take into account *re-rankings* of individual income receivers in the distribution as well as changes in inequality at a point in time. This intertemporal aspect of the multivariate problem focuses attention on the concept of *mobility* (Fields and Ok 1996, 1997) which may be useful in formalising important distributional concepts - such as the distinction between inequality of opportunity and inequality of outcome - that are logically separate from the issues on which we have focused in this chapter.

However there are other important interpretations of the same idea. In the analysis of personal taxation systems one is also interested in rerankings of individuals induced by the operation of the tax schedule as well as any equalisation of the distribution of income that the tax system may produce. (Berrebi and

[86]See Atkinson and Bourguignon (1982, 1987) and also the general discussion in Bourguignon (1989) and Ebert (1997c).

Silber 1983), (Ebert 1995b), (Jenkins and O' Higgins 1989), Lambert and Ramos 1997, Plotnick (1981, 1982, 1985), Silber (1995).

The way in which this class of problems may be addressed is to change the reference distribution from one of perfect equality to some given *status quo* distribution. So instead of the process

$$H^{(\bar{x})} \to F \tag{60}$$

where $H^{(\bar{x})}$ is a notional state of primordial equality at $\bar{x}$ and F is the actual income distribution, one considers the process

$$F_1 \to F_2 \tag{61}$$

where F_1 and F_2 can be the "before" and "after" income distributions in a historical process, or the pre- and post-tax distributions. Cowell (1980a) shows how the ideas here are related to Theil (1967)'s insights on information theory, and a general axiomatisation of this class of problems is provided in Cowell (1985b).

8 Empirical implementation

8.1 Some General Issues

The modern field of inequality measurement grew out of the intelligent application of quantitative methods to imperfect data in the hope of illuminating important social issues. The important social issues remain, and it is interesting to see the ways in which modern analytical techniques can throw some light on what it is possible to say about them.

A number of practical difficulties crop up throughout distributional analysis, some of which are common to applied statistics, and some which are associated with special problems that are characteristic of empirical income distributions: 8.1.1 to 8.1.3 highlight some of the main issues.

8.1.1 Data problems

Apart from the routine problems associated with data collection and interpretation several issues arise from the nature of the problem of comparing income distributions. Empirical income distributions typically have long tails with sparse data, and are possibly highly aggregated in order to protect confidentiality - this issue is discussed further in subsection 8.6. Difficulties with estimating what happens in the tails are further compounded by sensitivity of some estimation techniques and statistics to outliers and arbitrary truncation (Ben Horim 1990), (Nelson and Pope 1990), and to the presence of negative or zero values.[87] Be-

[87]This is basically a problem of the *a priori* specification of the support $\mathfrak{X}$: some interpretations of the "income" concept (expenditure, for example) explicitly rule out negative values, other interpretations (personal net worth, for example) allow them.

cause of the way in which data are collected[88] estimates of the values of inequality indices and other distributional tools may be subject to the impact of data contamination - see 8.2.2 below.

8.1.2 The questions to be addressed

It is clear from the previous sections that there is a variety of tools that are of potential interest to a researcher working on income inequality. The appropriate way in which to address a specific question on income distribution may be to employ a single index, a family of indices, or a more general ranking principle. The standard approach is to use sample statistics as estimates of the "true" population values of the various numerical tools (see subsection 8.2), and this strategy raises some basic questions about the type of test that will be appropriate when applying each of these types of tool. Clearly the empirical implementation of a single index that induces a complete order on $\mathfrak{F}$ is likely to be relatively straightforward in comparison to the implementation of a something like the Lorenz criterion which induces a ranking but leaves open the possibility of non-comparability of distributions - see page 14.

8.1.3 Modelling strategy

Perhaps the simplest, and most appealing, approach to the implementation of the distributional concepts introduced above is to use micro-data on incomes as though they represented a complete enumeration of a particular economy's income distribution. Using the raw data array in the form of (1) or (2) appears to yield a straightforward non-parametric approach to the analysis of income distribution. However to do this is to sweep aside a number of statistical difficulties which we will consider in subsection 8.2.[89]

An alternative approach to the problem is to develop an explicit model based on a parametric functional form. The functional form consists of two components:

- A general formula for a *class* of distribution functions $\mathfrak{F}_1' \subset \mathfrak{F}$.
- A parameter vector $\boldsymbol{\theta}$ which distinguishes one element $F_{\boldsymbol{\theta}} \in \mathfrak{F}'$ from another.

The empirical problem then consists of two parts: the specification of an appropriate functional form for the particular problem in hand, and then the estimation of the parameters of the selected functional forms according to appropriate statistical criteria. This is considered in subsection 8.7.

[88] It often happens that data providers will modify raw data so as to eliminate negatives or zeros (Jenkins 1997) or to censor high incomes on the grounds of confidentiality (Fichtenbaum and Shahidi 1988): this may amount to (well-meant) contamination of the data.

[89] It is also an unsatisfactory approach to the modelling of income distribution in the form of a density function (Figure 4); for discussion of the problems here see Cowell (1995), Chapter 5 and Appendix, and Silverman (1986)

8.2 Using sample data

The statistical approach to the subject exploits the analogy established between probability distributions and income distributions in 2.2 above. In this analogy we take X to be a random variable which is distributed according to $F \in \mathfrak{F}$ where X is "income" and x is a particular realisation of X: the problem is to relate this abstract construct to the concrete objects in a real-world data-set. Of course it is only rarely that a micro dataset represents a complete enumeration of an income distribution; typically we have to use a sample drawn from the distribution F; this is $F^{(n)}$, a distribution consisting of n point-masses, one at each observation in the sample. Fortunately several constructs that we have introduced in the abstract as appropriate tools for distributional analysis can be shown to be well-behaved when considering the relationship between the empirical representation and the underlying theoretical concept: for example it can be shown that the empirical (sample) Lorenz curve converges to the population Lorenz curve (Goldie 1977).[90]

8.2.1 Empirical measurement tools

The key concept that we require in empirical work is a *statistic*, which we have already met in other guises: in the case of univariate data a statistic is a functional T from $\mathfrak{F}$ to an appropriate range; for example, given the interpretation of the mean of a distribution in functional form (5), the sample mean will simply be $\mu\left(F^{(n)}\right)$. We may reinterpret other interesting entities used for distributional comparison - including inequality measures and ranking criteria - as statistics of a distribution. For a ranking or ordering $\succeq_T$ that embodies some given economic criterion T it is of particular interest to determine the properties of the corresponding statistic when applied to sample data.

8.2.2 Contamination and errors

It would be idle to suppose that a carefully constructed sampling procedure will resolve the main practical problems of empirical implementation. One may reasonably suppose that, because of misunderstanding, misrecording or misreporting, some of the observations are just wrong, and this may have a serious impact upon estimates of inequality measures (Van Praag et al. 1983). There are two principal types of approach to this problem, which find counterparts in the theoretical work of Sections 6 and 7 above.

The first of these can be illustrated by the elementary case depicted in Figure 18 where a mixture distribution has been constructed by combining the "true"

[90]There are several useful results of this sort. For example, if we consider the inequality statistic $I(F^{(n)})$, introduced in 8.2.1 below, then, from the Glivenko-Cantelli theorem, as $n \to \infty$, $I(F^{(n)})$ converges to $I(F)$ (Hoffman-Jø rgensen 1994b) page 105, (?). On the convergence of the Lorenz curve see also Csörgő and Zitikis (1995).

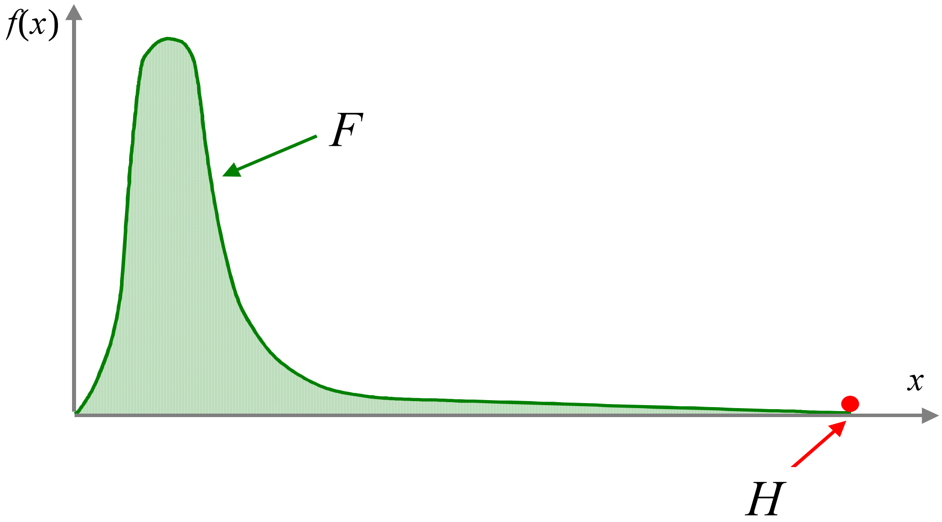

Figure 18: A mixture distribution

distribution F with an elementary point mass at income z (Cf. equation 8)

$$F_\delta^{(z)} = [1 - \delta] \, F + \delta H^{(z)} \tag{62}$$

as in the discussion of decomposability on page 12. The distribution $H^{(z)}$ represents a simple form of *data contamination* at point z, and δ indicates the importance of the contamination; $F_\delta^{(z)}$ is the observed distribution, and F remains unobservable.

Obviously if δ were large one would not expect to get sensible estimates of income-distribution statistics; but what if the contamination were very small? To address this question for any given statistic T one uses the *influence function* given by

$$IF(z; T, F) := \lim_{\delta \to 0} \left[\frac{T(F_\delta^{(z)}) - T(F)}{\delta} \right] \tag{63}$$

Then under the given model of data-contamination (62) the statistic T is *robust* if IF in (63) is bounded for all $z \in \mathfrak{X}$. The simple rule of thumb is that first-order dominance criteria and most poverty indices are indeed robust, whereas most inequality measures and second- and higher-order dominance criteria are not; with such inherently non-robust tools it is important that consideration be given to the treatment of zeros and outliers by one's estimation method.[91]

[91] See the discussion in 8.7.3 below and the results in Cowell and Victoria-Feser (1996a, 1996b, 1996c), Monti (1991) whose approach is based upon the work of Hampel (1968, 1974), Hampel et al. (1986), Huber (1986). Notice that the problem may sometimes be generated by the procedures involved in collecting data - see note 8.1.1 above.

The alternative approach is to consider that x is observed subject to measurement error. If this is so then presumably this will bias estimates of inequality (Chakravarty and Eichhorn 1994). An informal argument based on the decomposition by income source in subsection 7.2 illustrates this. What we actually observe is an income x which deviates from its "true" value $\tilde{x}$ thus:

$$x = \tilde{x} + v \tag{64}$$

where v is the realisation of a random variable that captures the effect of errors in measurement. It is clear that the error-model (64) has the same form as the source-decomposition problem in subsection 7.2: the relationship between "true" and "apparent" inequality can be deduced from a formula such as (55).

8.3 A standard class of inequality measures

To make effective use of sample data on income-distribution one should have a full specification of the sampling distribution of inequality measures and other tools of distributional analysis. A general treatment of the problem requires a full book, but some of the main issues can be illustrated by restricting attention to a few important special cases of inequality measurement.[92] In order to make the analogies with the relevant statistical literature more transparent let us modify our notation by introducing the following family of weighted moments about zero

$$\mu_{j,\eta}(F) := \int w^j x^\eta dF(w, x) \tag{65}$$

where w is used to allow for the possibility of population weights, as in (2) above. The GE class of inequality measures (21) can then be written[93]

$$I_{\mathrm{GE}}^{\alpha}(F) = \frac{\mu_{1,0}(F)^{\alpha-1}\mu_{1,1}(F)^{-\alpha}\mu_{1,\alpha}(F) - 1}{\alpha^2 - \alpha} \tag{66}$$

with appropriate limiting forms for the special cases $\alpha = 0, 1$.

8.3.1 Point estimates

To obtain the point estimates assume that a simple random sample $F^{(n)}$ has been drawn consisting of n observations (w_i, x_i), $i = 1, ..., n$, where x_i is the income

[92] For more details see Cowell (1999b).

[93] Note that in this notation moment $\mu_{1,0}$ can be interpreted as the "effective population size": if income-receivers are households and if the weight on each observation corresponds to the number of persons in each household, then $\mu_{1,0}(F)$ is exactly the number of persons in the population; if the weights are normalised by definition then $\mu_{1,0}(F) = 1$. Mean income is given by $\frac{\mu_{1,1}}{\mu_{1,0}}$.

and w_i the weight of observation i - Cf. the specification in equation (7).[94] Then the sample-moment counterparts to (65) are:

$$m_{j,\eta} := \mu_{j,\eta}\left(F^{(n)}\right) = \frac{1}{n}\sum_{i=1}^{n} w^j x^\eta \tag{67}$$

for any $j \in \{0, 1, 2\}$.

From (65) and (67) a consistent estimator of (66) is then given by

$$\frac{1}{\alpha^2 - \alpha}\left[\frac{m_{1,\alpha}}{m_{1,1}^{\alpha} m_{1,0}^{1-\alpha}} - 1\right] \tag{68}$$

This approach can easily be extended to inequality measures that are ordinally equivalent to (66), to other fully decomposable inequality measures and to measures that are ordinally equivalent to the form:

$$\int \phi\left(\frac{x}{\mu(F)}\right) dF \tag{69}$$

which covers most of the specific indices introduced earlier. However, the Gini coefficient is a little more problematic; perhaps its most convenient form for computation is as a sample version of (27) - a weighted sum of the ordered incomes:[95]

$$\sum_{i=1}^{n} \kappa[i]x[i], \tag{70}$$

where

$$\kappa[i] := \frac{w[i]}{m_{1,1}m_{1,0}}\left[2\sum_{j=1}^{i} w[j] - w[i] - 1\right], \tag{71}$$

and $(w[i], x[i])$ is the observation with the ith smallest x-value in the sample.

8.3.2 Inference

Now consider the problem of inference from micro data; for analytical convenience take first the class of scale-invariant, decomposable inequality measures. The basic result can be seen by examining the behaviour of $m_{1,\alpha}$, the sample estimate

[94] In the empirical distribution the weights usually play two roles: in addition to their use in reweighting the distribution by households to get the individual income distribution they may also incorporate sample weights: the weight for observation i is then $w_i = w_i' w_i''$ where w_i' is the ith observation's sampling weight and w_i'' is the household-to-individual weighting factor.

[95] Cf. Donaldson and Weymark (1980). On computational algorithms for the Gini see Berrebi and Silber (1987) , Lerman and Yitzhaki (1984, 1989), Shalit (1985) .

of $\mu_{1,\alpha}$, as an inequality measure which is essentially a non-normalised form of (66) with a sampling weight of unity for all observations.[96] We find:

$$
\begin{aligned}
var\left(m_{1,\alpha}\right) &= \frac{1}{n}\left[\mu_{2,2\alpha}-\mu_{1,\alpha}^{2}\right]. &(72)\\
\widehat{var}\left(m_{1,\alpha}\right) &= \frac{1}{n-1}\left[m_{2,2\alpha}-m_{1,\alpha}^{2}\right]. &(73)\\
\mathcal{E}\left(\widehat{var}\left(m_{1,\alpha}\right)\right) &= var\left(m_{1,\alpha}\right). &(74)
\end{aligned}
$$

From (72-74) we can obtain a result for all measures ordinally equivalent to the GE class - see (18) and (66). Define

$$
\boldsymbol{\mu} := \left(\mu_{1,0},\ \mu_{1,1},\ \mu_{1,\alpha}\right) \tag{75}
$$

and $\mathbf{m}$ as the sample counterpart of $\boldsymbol{\mu}$. Then the relevant class of inequality measures can be written as $\psi(\boldsymbol{\mu})$ and $\psi(\mathbf{m})$ in the population and the sample respectively (Cowell 1989, Thistle 1990). Given that:

$$
\sqrt{n}[\mathbf{m}-\boldsymbol{\mu}] \underset{n\to\infty}{\sim} N\left(0,\boldsymbol{\Sigma}\right), \tag{76}
$$

$$
\begin{aligned}
\boldsymbol{\Sigma} &: = \left[n\ cov(m_{1i},m_{1j})\right]_{i,j=0,1,\alpha}\\
&= \left[\begin{array}{ccc}
\mu_{2,0}-\mu_{1,0}^{2} & \mu_{2,1}-\mu_{1,0} & \mu_{2,\alpha}-\mu_{1,\alpha}\mu_{1,0}\\
\mu_{2,1}-\mu_{1,0} & \mu_{2,1}-\mu_{1,1}^{2} & \mu_{2,\alpha+1}-\mu_{1,\alpha}\mu_{1,1}\\
\mu_{2,\alpha}-\mu_{1,\alpha}\mu_{1,0} & \mu_{2,\alpha+1}-\mu_{1,\alpha}\mu_{1,1} & \mu_{2,2\alpha}-\mu_{1,\alpha}^{2}
\end{array}\right], \qquad (77)
\end{aligned}
$$

where N denotes the normal distribution (Rao 1973), we obtain as an asymptotic result:

$$
\sqrt{n}[\psi(\mathbf{m})-\psi(\boldsymbol{\mu})] \sim N\left(0,nV\right) \tag{78}
$$

where

$$
V := \frac{1}{n}\frac{\partial\psi^{T}}{\partial\boldsymbol{\mu}}\boldsymbol{\Sigma}\frac{\partial\psi}{\partial\boldsymbol{\mu}}, \tag{79}
$$

$$
\frac{\partial\psi}{\partial\boldsymbol{\mu}} := \left[\frac{\partial\psi(\boldsymbol{\mu})}{\partial\mu_{1,0}},\frac{\partial\psi(\boldsymbol{\mu})}{\partial\mu_{1,1}},\frac{\partial\psi(\boldsymbol{\mu})}{\partial\mu_{1,\alpha}}\right]^{T}. \tag{80}
$$

The quadratic form V in (79) is the asymptotic sampling variance of the inequality statistic, $\boldsymbol{\Sigma}$ is the variance-covariance matrix of the sample moments and $\psi_{\boldsymbol{\mu}}$ encapsulates the role of the inequality-cardinalisation in the sampling variance. In applying this result to the case of the GE class (in its standard cardinalisation 21) we find:

$$
\frac{\partial\psi}{\partial\boldsymbol{\mu}} = \frac{1}{\mu_{1,1}^{\alpha}\mu_{1,0}^{1-\alpha}}\left[\left[\alpha-1\right]\frac{\mu_{1,\alpha}}{\mu_{1,0}},\alpha\frac{\mu_{1,\alpha}}{\mu_{1,1}},1\right] \tag{81}
$$

[96] This is itself a valid non-normalised inequality measure for $\alpha > 1$ and $\alpha < -1$.

The bottom right-hand term in (77) would be the only relevant term if the data were unweighted and one had independent information about the true mean of the distribution. The neighbouring off-diagonal terms show that, if the mean is to be estimated from the sample, then its covariance with the income-evaluation function must be accounted for; likewise the remaining off-diagonal terms in illustrate the way individual weights are correlated with income (terms involving $\mu_{1,0}$) and with the income-evaluation function (bottom-left and top-right in the matrix): this correlation depends upon sample design and population heterogeneities which are inherent in inequality measurement. The variance of the inequality estimate in the case of weighted data could be larger or smaller than the corresponding variance in the unweighted case.

The normality of the sampling distribution (78) means that it is straightforward to apply standard statistical tests to problems involving distributional comparisons. For example a straightforward difference-of-means test could be applied to test whether inequality in one year was higher than that in another.[97]

8.4 Extensions

The methodology of subsection 8.3 can be extended to inequality indices that do not belong to the GE class such as the relative mean deviation (Gastwirth 1974a) although the formulae for the standard errors are not so neat.

It can also be applied to order statistics which form the basis for empirical implementation of the Lorenz curve concept and so also to the Gini coefficient:[98] ordinates of Lorenz curves (regular, generalised or absolute) are basically the sum of order-statistics, or a simple function of such sums. Consider the estimation problem in this case. In drawing such a curve one typically chooses a (finite) collection of population proportions $\Theta \subset [0,1]$ and then for each $q \in \Theta$ computes

$$\hat{c}_q := C\left(F^{(n)}; q\right) = \frac{1}{n} \sum_{i=1}^{\mathrm{int}(nq)} x_{[i]} \tag{82}$$

using (13) where $\mathrm{int}(z)$ denotes the largest integer less than or equal to z. The set of pairs $\{(q, \hat{c}_q/q) : q \in \Theta\}$ gives points on the empirical generalised Lorenz curve, and $\hat{c}_1$ is the sample mean $\mu\left(F^{(n)}\right)$; the relative and absolute curves are found by a simple normalisation as in (14) and (17). Under fairly mild conditions on the underlying distribution F the asymptotic distribution of the collection

[97] See (Zheng and Cushing 1996) for a discussion of tests on marginal changes in inequality.

[98] The underlying theory of the sampling distribution of order statistics was developed by Hoeffding (1948) - see also Moore (1968), Shorack (1972), Sillitto (1969). On the Gini coefficient see also Gastwirth and Gail (1985), Glasser (1962), Nygå rd and Sandström (1981), Sandström (1982, 1983), Sandström et al. (1985, 1988).

$\{\hat{c}_q\}$ is multivariate normal[99] (Beach and Davidson 1983) (Cowell and Victoria-Feser 1998) and so standard tests are available for comparing distributions,[100] and it is possible to construct confidence bands for Lorenz curves and associated tools (Anderson 1994). (Beach and Richmond 1985), (Csörgő and Zitikis 1996b, 1996a) . However this procedure raises a further issue: given that one wants to *rank* distributions according to some criterion T (in the manner of Section 3 - see definition 1) rather than simply *ordering* distributions according to a unique index I then there are two logical ways of testing for T-dominance of distribution G over distribution F with a sample from each distribution using a set of population proportions Θ: (1) the null hypothesis is $T(G;q) \geq T(F;q)$ for some $q \in \Theta$ and the alternative hypothesis is $T(G;q) < T(F;q)$ for all $q \in \Theta$, (2) the null hypothesis is $T(G;q) \geq T(F;q)$ for all $q \in \Theta$ and the alternative hypothesis is $T(G;q) < T(F;q)$ for some $q \in \Theta$; which approach is preferable depends on the significance level and the power of the test (Howes 1993).

8.5 Small sample problems

The asymptotic results in 8.3.2 may not be valid for some empirical applications for reasons that are readily apparent. First, it is often the case that the particular problem of economic interest requires a subsample that is of fairly modest size: the sample or subsample may be so small that the asymptotic results which are commonly invoked are invalid. The assumption is sometimes made that sample data on income distribution will, of their nature, have a large n so that in practice the issue of sampling error can be neglected as being of secondary importance. Second, some particularly sensitive indices (for example the coefficient of variation) have a standard error of estimate that it is very large even for apparently

[99]Beach and Davidson (1983) assume that F is twice differentiable. For any $q, q' \in \Theta$ such that $q \leq q'$ then the asymptotic covariance of $\sqrt{n}\hat{c}_q$ and $\sqrt{n}\hat{c}_{q'}$ is

$$n \left[q\sigma_q^2 + [qx_q - c_q] \left[x_{q'} - q'x_{q'} + c_{q'} - \frac{c_q}{q} \right] \right]$$

where x_q and c_q are the population quantiles and income cumulants

$$\begin{aligned} x_q &:= Q(F;q) \\ c_q &:= C(F;q) \end{aligned}$$

(see equations 12 and 13), and σ_q^2 is the conditional variance

$$\tfrac{1}{q} \int^{Q(F;q)} x^2 dF(x) - \left[\tfrac{c_q}{q} \right]^2 .$$

[100]For applications of classical hypothesis testing to ranking criteria see, for example, Beach et al. (1994), Bishop et al. (1987, 1988, 1989, 1994), Bishop et al. (1991), Bishop et al. (1989), Bishop et al. (1997), Davidson and Duclos (1997), Stein et al. (1987), (Zheng 1996).

large samples. Under these circumstances it may be appropriate to use statistical methods which involve resampling with ("the bootstrap") or without ("the jackknife") replacement using the empirical distribution $F^{(n)}$ as raw materials.[101]

8.6 The problem of grouped data

Even though micro-data are widely available today it is still necessary to work with grouped data on income distributions; if one wishes to examine historical data then the issue of grouping is almost unavoidable . Typically we have the following situation. The structure of the data imply that there is a partially specified income distribution of the form:

$$F(x) = \sum_{i=1}^{j} p_i, \ x = a_{j+1}, j = 1, 2, ..., J - 1 \tag{83}$$

However, detailed information on the distribution within each interval $[a_j, a_{j+1})$ is usually unavailable except that one sometimes has the information

$$\frac{1}{p_j} \int_{a_j}^{a_{j+1}} x dF(x) = \bar{x}_j \ , \tag{84}$$

the mean of the empirical distribution within interval j. The situation is as though one had a partition of the nonoverlapping form described in 6.2.2 with limited information about the within-group income distributions $F^{(j)}$. Furthermore, not only is information unavailable about the shape of the distribution in the top income interval $[a_J, a_{J+1})$, but the value of a_{J+1} is also usually left unspecified.

It is clear that this information structure will result in the loss of some important distributional information (Howes 1996) and will complicate some standard statistical problems such as inference - see Gastwirth et al. (1986), Gastwirth et al. (1989). However there are a number of additional problems - special to this type of data: - concerning the possibility of an appropriate assumed distribution $\tilde{F}^{(j)}$ in interval j:

- Is it possible to choose $\tilde{F}^{(j)}$ so as to put *bounds* on inequality estimates in the light of the partial information?

[101] Although bootstrap estimates have a smaller sampling variance than their jackknife counterparts, they are usually much more time-consuming computationally. For a discussion of the bootstrap and jackknife approaches see Bhattacharya and Qumsiyeh (1989), (Csörgő and Mason 1989), Efron (1979, 1982), Efron and Tibshirani (1993), Hall (1982), Rubin (1981), Shao and Tu (1995); see also Kish and Frankel (1970) for a discussion of cases where samples are complex. However, the bootstrap does not work in every case, especially when - as with relative Lorenz ordinates - the statistic to be bootstrapped is bounded; see Schenker (1985) and Andrews (1997). For an application of the bootstrap and jackknife to inequality statistics see Efron and Stein (1981), Maasoumi et al. (1997), Mills and Zandvakili (1997), Yitzhaki (1991).

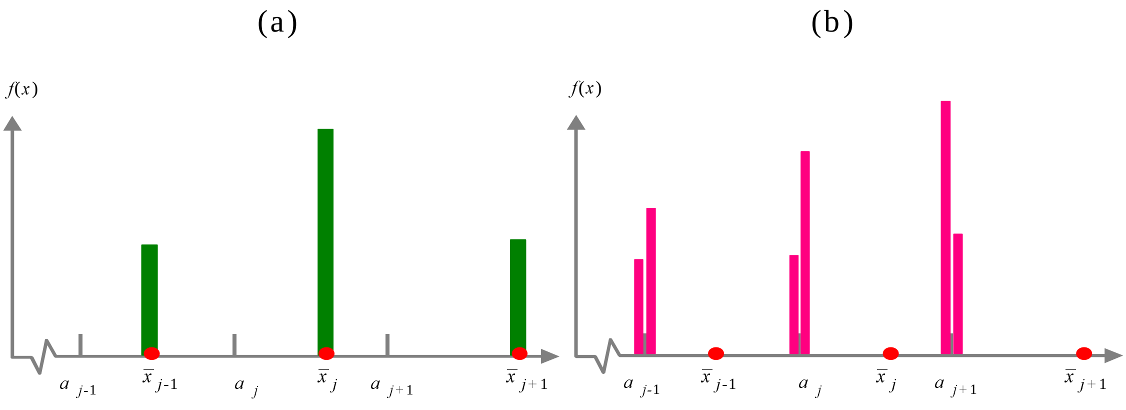

Figure 19: Assumed distributions for (a) lower-bound inequality, (b) upper-bound inequality

- Are there appropriate interpolation methods to fit *compromise* estimates $\tilde{F}^{(j)}$ of the underlying distribution?
- How is one to choose $\tilde{F}^{(J)}$ so as to handle the problem of the open-ended interval $[a_J, \infty)$?[102]

8.6.1 The bounding problem

Unsophisticated methods of choosing $\tilde{F}^{(j)}$ to obtain bounding values of inequality measures are easily obtained. Given the information (83),(84) the lower bound on inequality is found by assigning a point mass p_j at point $\bar{x}_j$ in each closed interval j, and the upper bound is found by assigning a point mass λp_j at the lower boundary a_j and a point mass $[1-\lambda]p_j$ at the upper boundary a_{j+1} of each closed interval j where $\lambda := \frac{a_{j+1}-\bar{x}_j}{a_{j+1}-a_j}$ - see Figure 19.

If it is legitimate to make more specific assumptions about $F^{(j)}$, the unknown distribution within a particular interval $[a_j, a_{j+1})$, namely that there is decreasing frequency over that interval, then it is possible to obtain more refined bounds on the values of measures that have the additively separable form (36). For the refined lower-bound value of inequality the assumed distribution $\tilde{F}^{(j)}$ within the interval is assumed to be rectangular over the sub-interval $[a_j, 2\mu_j - a_j)$ and zero elsewhere; the refined upper-bound case is found by supposing $\tilde{F}^{(j)}$ to consist of a point mass $p_j[a_{j+1} + a_j - 2\bar{x}_j][a_{j+1} - a_j]^{-1}$ at a_j and a rectangular distribution $2p_j[\bar{x}_j - a_j][a_{j+1} - a_j]^{-2}$ over (a_j, a_{j+1}) - see Figure 20 (Gastwirth 1975). Other bounding results are obtainable for alternative assumptions about the amount of

[102] In some cases a_1 is left unspecified so that a similar problem arises with $\tilde{F}^{(1)}$.

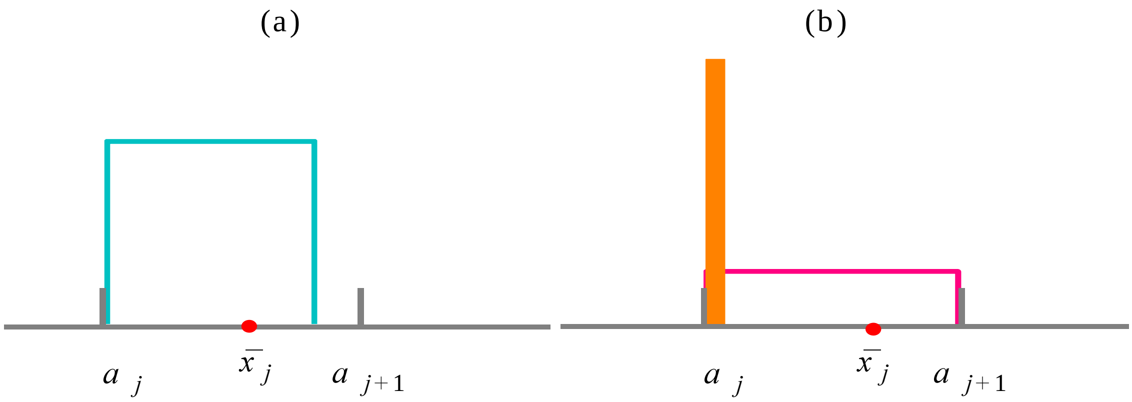

Figure 20: Assumed distribution for refined bounds: (a) lower, (b) upper

information available about the grouped income data (Cowell 1991), (Gastwirth 1972), (McDonald and Ransom 1981), (Murray 1978).

8.6.2 Interpolation methods

If one were trying to choose $\tilde{F}^{(j)}$ to obtain a compromise estimate of inequality what type of restrictions on the distribution would it be reasonable to impose? It might be that essentially aesthetic properties - such as smoothness or flexibility of form of $\tilde{F}^{(j)}$, or continuity of the implied density across interval boundaries, or familiarity of the functional form used in interpolation - appear as particularly attractive. However relatively simple interpolation schemes - such as those depicted[103]- actually perform as well as more elegant formulae (Cowell and Mehta 1982), (Gastwirth and Glauberman 1976). What is far more important than the detail of the interpolation formula is the fineness of the information provided by the data source.[104]

8.6.3 The upper tail

Unbounded intervals are convenient for information providers but present a difficult problem for the distributional analyst. Although in some cases it is possible to use the data to impose an upper bound on inequality estimates (see the references in 8.6.1) the standard approach is to model the distribution in the top one

[103]In Figure 21(a) the interpolation formula is piecewise linear with discontinuities at the interval boundaries a_j; in Figure 21(b) the interpolation formula is piecewise rectangular in the subintervals $[a_j, \bar{x}_j)$ and $[\bar{x}_j, a_{j+1})$ - see Cowell and Mehta (1982) for details.

[104]The question of optimal grouping by data providers is discussed by Aghevli and Mehran (1981), Davies and Shorrocks (1989).

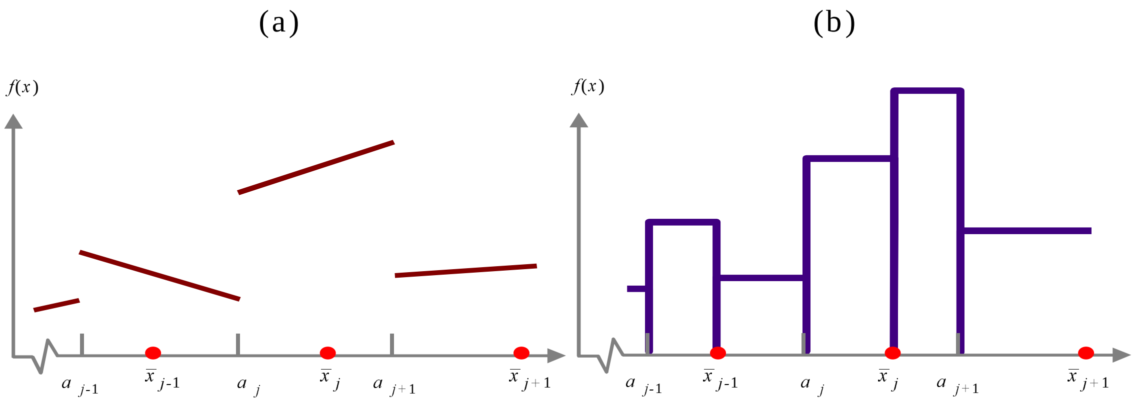

Figure 21: (a) Linear and (b) Split-histogram interpolation

or two intervals using an explicit functional form, usually a Pareto distribution (Needleman 1978), (Fuller 1979) - see subsection 8.7.

8.7 Modelling with functional forms

The use of the parametric approach to distributional analysis runs counter to the general trend towards the pursuit of non-parametric methods, although is extensively applied in the statistical literature. Perhaps it is because some versions of the parametric approach have had a bad press: Pareto's seminal works led to some fanciful interpretations of "laws" of income distribution (Davis 1941, 1954); perhaps it is because the non-parametric method seems to be more general in its approach.[105]

Nevertheless a parametric approach can be particularly useful for estimation of inequality indices or other statistics in cases where information is sparse (Braulke 1983, 1988), (Chotikapanich et al. 1997). Furthermore some standard functional forms claim attention, not only for their suitability in modelling some features of many empirical income distributions but also because of their role as equilibrium distributions in economic processes.

8.7.1 The choice of functional form

In the inequality literature there is a substantial number of formulae $F_{\boldsymbol{\theta}}$ used to model various aspects of income distribution, most of which have natural intuitive interpretations of the parameters $\boldsymbol{\theta}$. Some of the most important include the following:

[105] Although for some issues, such as kernel density estimation, specialised structural assumptions are required (Silverman 1986).

- the Pareto model (Arnold 1983), (Chipman 1974),

$$F_{\underline{x},\alpha}(x) = 1 - \left[\frac{\underline{x}}{x}\right]^{\alpha} , \tag{85}$$

where $\underline{x} > 0$ is a location parameter and α is a parameter that is inversely related to dispersion.

- the lognormal model (Aitchison and Brown 1954, 1957)

$$F_{m,\sigma^2}(x) = \int_0^x \frac{1}{y\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}[\log(y)-m]^2} \, dy , \tag{86}$$

where m and σ^2 are parameters specifying the mean and variance of log-income.

- the gamma model (Salem and Mount 1974)

$$F_{\alpha,\lambda}(x) = \int_0^x \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y} \, dy \tag{87}$$

where Γ is the standard gamma function.

The functional form that is appropriate for modelling distributions depends on the definition of income and the particular part of the distribution in which we happen to be interested. For example, the Pareto model is appropriate for analysing upper incomes; in general lognormal models are appropriate for individual earnings in a homogeneous population; the gamma model approximates the majority of data in the "middle" of the distribution.(Harrison 1981)

The families of two-parameter models (85)-(87) are evidently limited in the variety of shapes of income distributions that they can be expected to describe. One way forward is to consider extensions to the basic forms to make them more flexible - for example multi-parameter generalisations of the Pareto and of the lognormal have been suggested.[106] Several other families of distributions have been shown to have merit in capturing some important features of the distribution; many of these functional forms are interrelated, in the sense that one is a special form of another, or one approximates another asymptotically.[107] But it should be borne in mind that, however attractive greater flexibility may seem to be, proliferation of parameters in the model specification may impose a considerably greater burden in terms of interpretation and computation. Four

[106]See for example Rasche et al. (1980), Gupta (1984), Rao and Tam (1987), Singh and Maddala (1976) on Pareto-type indices, Esteban (1986), Kloek and Van Dijk (1977, 1978), Taille (1981) on the generalised gamma, Metcalf (1969) on the generalised lognormal, and also Cowell 1995, Appendix for a discussion of these alternative forms.

[107]See McDonald (1984), page 648 and Merkies (1994) for a useful diagram showing the principal family connections. See also Majumder and Chakravarty (1990).

parameter models can be very unwieldy, and even three-parameter models may not give a huge advantage over their two-parameter counterparts. Complicated empirical distributions may not be much illuminated by complicated functional forms: it may be better to piecemeal focus on readily interpretable chunks of the distribution.

8.7.2 Inequality in parametric models

In most cases the use of a functional form induces a structure on $\mathfrak{F}'$ that makes the distributional ranking problem very easy - perhaps deceptively so. For example the Lorenz curves of Pareto distributions never intersect; the same is true for lognormal distributions (the Lorenz curve lies further from the line of equality the lower is the parameter α and the higher is the parameter σ^2 in (85) and (86) respectively); so adoption of either of these families as a paradigm for the admissible class of distributions means that first- and second- order dominance criteria are always very clear. Of course one has to be careful that one is not squeezing a foot into an ill-fitting standardised shoe: unambiguous dominance results are of little use if the adoption of the functional form is at the price of ignoring important information in some part of the distribution.

If one does adopt a specific functional form, then inequality can be expressed in terms of its parameters: [108]

$$I(F_{\boldsymbol{\theta}}) = \iota(\boldsymbol{\theta}) \tag{88}$$

For example the Atkinson family of indices (30) can be written

$$I_{\mathrm{A}}^{\varepsilon}\left(F_{\underline{x},\alpha}\right) = 1 - \frac{\alpha-1}{\alpha}\left[\frac{\alpha}{\alpha+\varepsilon-1}\right]^{\frac{1}{1-\varepsilon}} \tag{89}$$

and

$$I_{\mathrm{A}}^{\varepsilon}\left(F_{m,\sigma^2}\right) = 1 - e^{-\frac{1}{2}\varepsilon\sigma^2} \tag{90}$$

in the case of the Pareto and the lognormal respectively. Again note that inequality is monotonic decreasing in α (89) and monotonic increasing in σ^2 (90) respectively.

8.7.3 The estimation method

The general nature of the problem can be described as follows. Firstly it is necessary to choose a functional form or model that is appropriate in an economic sense - i.e. a general family form $F_{\boldsymbol{\theta}}$ that captures the general shape of the distribution, or part of the distribution, that one wishes to model, as described

[108] See for example Cowell (1995), Chapter 4, Wifling (1996), Wifling and Krämer (1993); also Aitchison and Brown (1957) for the lognormal and Chipman (1974) for the Pareto.

in 8.7.1 above. Then the model parameters are estimated using an appropriate algorithm: this means an algorithm chosen according to criteria which implicitly define the term "appropriate" in the statistical sense.

For example, "appropriateness" is often interpreted in terms of efficiency of the estimator: given a model $F_{\boldsymbol{\theta}}$ with density function $f_{\boldsymbol{\theta}}$, the *maximum likelihood estimators* (MLE) are then obtained as the solution in $\boldsymbol{\theta}$ of the m equations[109]

$$\sum_{i=1}^{n} \mathbf{S}(x_i; \boldsymbol{\theta}) = 0 \tag{91}$$

where $m := \dim(\boldsymbol{\theta})$ and $\mathbf{S}$ is the scores function defined by

$$\mathbf{S}(x; \boldsymbol{\theta}) = \frac{\partial}{\partial \boldsymbol{\theta}} \log f_{\boldsymbol{\theta}}(x) \tag{92}$$

Of course it is clear that this efficiency criterion cannot take account of the problem of contamination mentioned in 8.2.2 above. It may be realistic to assume that the data come from a distribution in the neighbourhood of the true model of the distribution - that they are actually generated by the parametric model $F_{\boldsymbol{\theta}}$ with probability $1 - \delta$, and by an alien distribution (the contamination) H with probability δ, where δ is small. The MLE procedures would be optimal given the assumption that the data are generated by $F_{\boldsymbol{\theta}}$ (the case $\delta = 0$), but will be invalid for any variation around $F_{\boldsymbol{\theta}}$ (the case $\delta > 0$) - see Hampel et al. (1986), and Victoria-Feser (1993). To handle this requires an alternative statistical criterion of appropriateness that takes into account the robustness considerations outlined in 8.2.2. In the robust approach to estimation, instead of applying (91 and 92) one requires an algorithm to filter outlying observations systematically. The MLE belong to a general class of so-called *M-estimators* which are defined as the solution in $\boldsymbol{\theta}$ of

$$\sum_{i=1}^{n} \psi(x_i; \boldsymbol{\theta}) = 0 \tag{93}$$

where ψ belongs to a very general class of functions. (Huber 1964) The robust approach consists of a search for the minimum (asymptotic) variance M-estimator with a bounded IF: efficiency is sacrificed to some extent in favour of robustness. There is a number of optimal estimators, depending on the exact method of bounding the IF.[110]

[109] For the MLE in the case of the Pareto see Baxter (1980); and for the MLE for a variety of functional forms see McDonald and Ransom (1979).

[110] For example consider the standardised *Optimal Bias-Robust Estimators (OBRE)* which also belong to (93); given a constant $c \in [\sqrt{m}, \infty)$ which plays the role of upper bound on the IF, the OBRE is defined as the solution in $\boldsymbol{\theta}$ to

$$\textstyle\sum_{i=1}^{n} \psi_c^{\mathbf{A},\mathbf{a}}(x_i; \boldsymbol{\theta}) = 0$$

Other criteria of appropriateness could be applied. For example the *method of moments* has the advantages of simplicity and relative transparency: by equating the theoretical values, conditional on $\boldsymbol{\theta}$, of some of the moments to their counterparts computed from the data one can obtain a set of simultaneous equations for the explicit computation of the parameter estimates. How many moments should be used, and which ones, will of course depend on the particular form of $F_{\boldsymbol{\theta}}$; but typically one sets up two equations in the mean and variance of income or log-income.

8.8 Re-using inequality measures

The computation of inequality in the case of parametric approaches to income distribution has suggested a number of alternative applications of measurement tools. Prominent among these are instances where the inequality measures are "turned on their heads" and used as the basis of goodness-of-fit tests (Gail and Gastwirth 1978a, 1978b) or devices for quantifying the distance of one distribution from another (Atkinson et al. 1988). The inequality index is used to give meaning to the deviations between the sample data and a particular functional form proposed as a model of the underlying income distribution.

Applying inequality measures in this sort of way continues to be a promising idea. For some time distributional tools developed for the purpose of inequality analysis have been applied in contexts other than income distribution - such as in the study of industrial concentration, political science, for example. It seems appropriate that these measurement techniques which have been underpinned by careful analytical work on their axiomatic bases and their structural properties should be applied in contexts other than a narrow welfare-economic interpretation of inequality. An example of this is the development of the concept of the

---

where

$$\begin{aligned} \psi_c^{\mathbf{A},\mathbf{a}}(x;\boldsymbol{\theta}) &= \mathbf{A}(\boldsymbol{\theta})\left[\mathbf{S}(x;\boldsymbol{\theta}) - \mathbf{a}(\boldsymbol{\theta})\right] w_c(x;\boldsymbol{\theta}) \\ w_c(x;\boldsymbol{\theta}) &= \min\left\{1; \frac{c}{\|\mathbf{A}(\boldsymbol{\theta})\left[\mathbf{S}(x;\boldsymbol{\theta}) - \mathbf{a}(\boldsymbol{\theta})\right]\|}\right\} \end{aligned}$$

$\mathbf{A}$ is an $m \times m$-matrix, and $\mathbf{a}$ is an m-vector; $\mathbf{A}$ and $\mathbf{a}$ are determined by:

$$\begin{aligned} E\left[\psi_c^{\mathbf{A},\mathbf{a}}(x;\boldsymbol{\theta})\psi_c^{\mathbf{A},\mathbf{a}}(x;\boldsymbol{\theta})^T\right] &= \mathbf{I} \\ E\left[\psi_c^{\mathbf{A},\mathbf{a}}(x;\boldsymbol{\theta})\right] &= \mathbf{0} \end{aligned}$$

$\mathbf{A}$ and $\mathbf{a}$ can be considered as Lagrange multipliers for these last two restrictions ; ψ is a modified and standardised scores function, weighted using w_c. The constant c may be selected as a "regulator" between the two statistical criteria, efficiency and robustness. Lower values of c yield more robust, but less efficient, estimators: the maximum-robustness estimator corresponds to the lower bound of the constant $c = \sqrt{m}$; on the other hand $c = \infty$ yields the MLE (Prieto-Alaiz and Victoria-Feser 1996) (Victoria-Feser 1995). On robust estimation for grouped data see Victoria-Feser and Ronchetti (1997)

"Gini regression" technique (Olkin and Yitzhaki 1992), (Yitzhaki 1996). Re-using inequality measures in this way can provide important insights in other fields of economics and gives the prominence to the analysis that it justly deserves.

9 A Brief Conclusion

A conclusion section provides an ideal opportunity to pass judgment on dead ends and promising discoveries, and to make hopeful remarks about further work in the subject. I shall not try such grand things: instead, let us briefly consider the natural limitations of the subject as it is currently understood.

The formal approach to inequality measurement is a good discipline for training us in thinking about what we are doing when income distributions are to be compared. It does not matter much what the ethical or intellectual standpoint is from which one comes to the subject; the analysis can assist in providing systematic answers to a number of basic questions. Is inequality about individual incomes or about income differences? Is the shape of the income distribution relevant to inequality judgments? How can theoretical reasoning in the economics discipline illuminate practical questions of economic and social policy?

On the other hand some major issues are sidestepped or are deliberately left open for work by scholars in related fields: Why *should* social welfare be concerned with inequality? Does inequality "matter" in a social sense? How should inequality be related to broader concepts of "fairness" in economics?

Finally, an odd question. Why is something like the Gini coefficient so consistently popular? A trawl through the empirical literature (which I shamefully neglected in this survey) reveals an overwhelming propensity to use this index rather than other tools reviewed above. For many practical people doing important applied work it is *the* inequality index. Yet the index has many apparent drawbacks. It is not decomposable, at least not in the sense that it will satisfy consistency requirements for arbitrary partitions of the population. Its statistical properties are far less tractable than those of easily available alternatives. It does not emerge naturally from the welfare-economics of the subject (though it can be made to fit in). It is certainly not new. Here are two possible answers, both of which are offered tentatively:

- There is also considerable cultural inertia in the field of inequality analysis, as in other fields.
- Perhaps it is because people can "see" inequality immediately once the idea of the Lorenz curve is accepted.

Each of these reasons may be no bad thing. However the question, and its possible answer, may illuminate some of the problems that academic researchers in this field could have in connecting theory with an understanding of the real world.

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