

Inequality:  
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Cowell,  
Flachaire

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# Measuring Inequality with ordinal data

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Summary

- Inequality analysis: origins go back to Pigou and Dalton
  - explicitly tied into welfare: contrast Gini and Lorenz
  - seen as more fundamental than approaches such as Pareto
- But all of this is erected on rather demanding informational structure
  - income, wealth, cardinally measurable and comparable
  - income, earnings usually assumed to be non-negative
- Maybe need a new approach to inequality measurement

# Ingredients of Measurement Problem

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Summary

- 3 ingredients of the income-inequality measurement problem:
  - the definition of “income”
  - the definition of the “income-receiving unit”
  - method of aggregation
- Same issues arise in cases where “income” is ordinal
- Look at standard income-inequality problem before modelling ordinal-data problem

# Income Inequality

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Summary

- 3 ingredients:
  - **“income”**: family income, earnings, wealth  $x \in X \subseteq \mathbb{R}$ .
  - **“income-receiving unit”**:  $n$  persons
  - **method of aggregation**: function  $X^n \rightarrow \mathbb{R}$
- Usually work with  $X_\mu^n \subset \mathbb{R}$
- $X_\mu^n$ : Distributions obtainable from a given total income  $n\mu$  using lump-sum transfers
- Obviously can't do that here:  $\mu$  is undefined

# Utility (1)

## Cardinalisation and inequality

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Summary

- 3 ingredients:
  - **“income”**:  $u = U(x)$ .
  - **“income-receiving unit”**:  $n$  persons (as before)
  - **method of aggregation**: function  $\mathbb{U}^n \rightarrow \mathbb{R}$
- Problem of cardinalisation
- But just assuming cardinal utility is no use
  - Already pointed out in Atkinson (1970)
  - Dalton (1920) suggested inequality of (cardinal) utility
  - But if, for all  $i$ , you multiply  $u_i$  by  $\lambda \in (0, 1)$  and add  $\delta = \mu[1 - \lambda] \dots$
  - ...this will automatically reduce measured inequality.
- Is this just a technicality?
- Can we proceed just as with regular income?

# Utility (2)

Is this something different?

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- Atkinson and Dalton examples of “aggregation process”
  - How social values are introduced into an inequality-evaluation of income distribution...
  - ...not the inequality-evaluation of a distribution of utilities.
- Sometimes these are equivalent
  - but sometimes not
  - maybe utility has no natural income equivalent?
- Case 1.  $U$  depends on  $x$  with no agreed monetary valuation
  - quality of life
  - happiness
- Case 2.  $U$  depends on  $x$  that is categorical:
  - health status
  - level of completed education
  - access to public services

# Categorical variable

Example: Access to Services

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	<u>Case 1</u>	<u>Case 2</u>
	$n_k$	$n_k$
<u>B</u> oth Gas and Electricity	25	0
<u>E</u> lectricity only	25	50
<u>G</u> as only	25	50
<u>N</u> either	25	0

- Suppose we have no information about needs / usage
- Nevertheless it is clear that Case 1 seems more unequal than Case 2



# Ways Forward?

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- We could try to develop dominance criteria based on median
  - Median may be well defined although mean is not
  - what principle should play the role that is played by PoT in income inequality?
- Could try a family of measures using only median
- For such things as happiness could just use arbitrary cardinalisation
  - over large part of domain may be empirically robust
  - psychologists think Likert scales are OK for cardinalising
  - but what happens in tails?

# Status and Information

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Summary

- Step 1 is to define status
  - depends on the purpose of inequality analysis
  - depends on structure of information
  - conventional inequality approach only works in narrowly defined information structure
- In some cases a person's status is self-defining
  - income
  - wealth
- In some cases status is defined given additional distribution-free information
  - example: if it is known that utility is  $\log(x)$
- In some cases status requires information dependent on distribution
  - GRE
  - TOEFL

# Status and Distribution (1)

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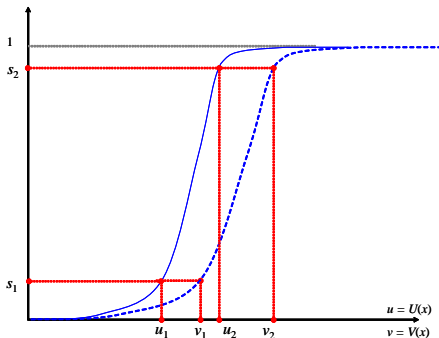
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Summary

- $i$ 's status uniquely defined for a given distribution of  $u$



- disposes of the problem of cardinalisation
  - $U$  and  $V = \varphi(U)$  two cardinalisations of the utility of  $x$
  - for each  $i:u_i$  and  $v_i$  map into  $s_i$

# Status and distribution (2)

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- This approach works for categorical data
  - we just have an ordered arrangement of categories  $1, 2, \dots, k, \dots, K$
  - and the numbers in each category  $n_1, n_2, \dots, n_k, \dots, n_K$
- Merger principle
  - merge two adjacent categories that are irrelevant for  $i$
  - then this should leave  $i$ 's status unaltered
- Merger principle implies that  $s$  should be additive in the  $n_k$ 
  - could have upward-looking ...
  - ... or downward-looking status

# Elements of the Model

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Summary

- individual's status is given by  $s \in S \subseteq \mathbb{R}$ 
  - status determined from utility using  $\psi$
- vector of status in a population of size  $n$  :  $\mathbf{s} \in S^n$ .
- $e \in S$  : an equality-reference point
  - could be specified exogenously
  - could also depend on status vector  $e = \eta(\mathbf{s})$
  - $\eta$  need not be increasing in each component of  $\mathbf{s}$
- Inequality: aggregate distance from  $e$ 
  - don't need an explicit distance function
  - implicitly define through inequality ordering  $\succsim$

# Basic Axioms

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Summary

- **[Continuity]**  $\succeq$  is continuous on  $S^n$ .
- **[Monotonicity in distance]** If  $\mathbf{s}, \mathbf{s}' \in S_e^n$  differ only in their  $i$ th component then (a) if  $s'_i \geq e : s_i > s'_i \iff \mathbf{s} \succ \mathbf{s}'$ ; (b) if  $s'_i \leq e : s'_i > s_i \iff \mathbf{s} \succ \mathbf{s}'$ .
- **[Independence]** For  $\mathbf{s}, \mathbf{s}' \in S_e^n$ , if  $\mathbf{s} \sim \mathbf{s}'$  and  $s_i = s'_i$  for some  $i$  then  $\mathbf{s}(\zeta, i) \sim \mathbf{s}'(\zeta, i)$  for all  $\zeta \in [s_{i-1}, s_{i+1}] \cap [s'_{i-1}, s'_{i+1}]$ .
- **[Anonymity]** For all  $\mathbf{s} \in S^n$  and permutation matrix  $\mathbf{P}$ ,  $\mathbf{P}\mathbf{s} \sim \mathbf{s}$ .

# Standard result

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## Theorem

*Continuity, Monotonicity, Independence, Anonymity jointly imply  $\succeq$  is representable by the continuous function  $I : S_e^n \rightarrow \mathbb{R}$  where  $I(\mathbf{s}; e) = \Phi(\sum_{i=1}^n d(s_i, e), e)$ , where  $d : S \rightarrow \mathbb{R}$  is a continuous function that is strictly increasing (decreasing) in its first argument if  $s_i > e$  ( $s_i < e$ ).*

## Corollary

*Inequality is total “distance” from equality. Distance  $d$  is continuous, satisfies  $d(e, e) = 0$ .  $d(s, e)$  is increasing in status if you move away from the reference point.*

# Structure Theorem

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Summary

- We need to impose more structure on the problem
- **[Scale irrelevance]** For all  $\lambda \in \mathbb{R}_+$ : if  $\mathbf{s}, \mathbf{s}' \in S_e^n$  and  $\lambda \mathbf{s}, \lambda \mathbf{s}' \in S_{\lambda e}^n$  then  $\mathbf{s} \sim \mathbf{s}' \Rightarrow \lambda \mathbf{s} \sim \lambda \mathbf{s}'$ .

## Theorem

*Impose also Scale irrelevance. Then  $\succeq$  is representable by  $I(\mathbf{s}; e) = \Phi(\sum_{i=1}^n d(s_i, e), e)$ , where the function  $d$  takes the form  $(s, e) = e^c \phi\left(\frac{s}{e}\right)$ ,  $\phi$  is a continuous function and  $c$  is an arbitrary constant.*



# Characterisation Theorem

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- We now impose yet more structure on the problem

- **[Ratio scale irrelevance]** Suppose there are  $\mathbf{s} \in S_e^n$  and  $\mathbf{s}^\circ \in S_{e^\circ}^n$  such that  $\mathbf{s} \sim \mathbf{s}^\circ$ . Then for all  $\lambda > 0$ ,  $\mathbf{s}' \in S_{e'}^n$  and  $\mathbf{s}'' \in S_{e''}^n$ , such that for each  $i$ ,  $s'_i/e' = \lambda s_i/e$  and  $s''_i/e'' = \lambda s_i^\circ/e^\circ$ :  $\mathbf{s}' \sim \mathbf{s}''$ .

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## Theorem

*Impose also Ratio scale irrelevance. Then  $\succeq$  is representable as  $\Phi(I(\mathbf{s}; e), e)$  where  $I_\alpha(\mathbf{s}; e) = \frac{1}{\alpha[\alpha-1]} \left[ \frac{1}{n} \sum_{i=1}^n \left[ \frac{s_i}{e} \right]^\alpha - c \right]$ ,  $\alpha, c \in \mathbb{R}$  and  $\Phi$  is increasing in its first argument.*

- Gives a family of measures, contingent on  $e$  and  $c$

# A Usable Inequality Index?

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Summary

- Class of functions that could be used as inequality measures:
  - $\Phi(I(\mathbf{s}; e), e)$
  - $e = \eta(\mathbf{s})$ , the reference point
  - $I(\mathbf{s}; e) = I_\alpha(\mathbf{s}; \eta(\mathbf{s})) = \frac{1}{\alpha[\alpha-1]} \left[ \frac{1}{n} \sum_{i=1}^n \left[ \frac{s_i}{\eta(\mathbf{s})} \right]^\alpha - c(\eta(\mathbf{s})) \right]$
- Key questions:
- Do functions of the form  $\Phi(I(\mathbf{s}; e), e)$  “look like” inequality measures?
  - transfer principle?
  - reference point?
  - sensitivity to parameters
- What is the appropriate form for  $\Phi$  ?
  - may depend on the reference status  $e$
  - may depend on interpretation

# Transfer Principle (1)

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Summary

- Standard version of transfer principle is not applicable
  - “Mean status” is not quite like mean income
  - can change in interesting ways
- Can show a property related to transfer principle
  - if  $e$  is independent of  $s$
  - or if  $e$  depends only on  $\mu(s) = \frac{1}{n} \sum_{i=1}^n s_i$
- Then for all  $\alpha$  in such cases:
  - if  $i$ 's status increases  $\delta > 0$  and  $j$ 's status decreases by  $\delta$
  - such that  $s_i < s_j$  and  $s_i + \delta < s_j - \delta$ ,
  - then inequality is reduced
- But is this property attractive?

# Four distributional scenarios (1)

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		Case 0		Case 1		Case 2		Case 3	
		$n_k$	$s_i$	$n_k$	$s_i$	$n_k$	$s_i$	$n_k$	$s_i$
Motivation	<b>B</b>	0		25	1	0		25	1
Basic Problem	<b>E</b>	50	1	25	3/4	50	1	25	3/4
Previous work	<b>G</b>	25	1/2	25	1/2	50	1/2	50	1/2
Approach	<b>N</b>	25	1/4	25	1/4	0		0	
Model									
Basic structure									
Characterisation									
Inequality Measures	$\mu(\mathbf{s})$		11/16		5/8		3/4		11/16

- $n_k$  is # persons in category  $k \in \{\mathbf{B}, \mathbf{E}, \mathbf{G}, \mathbf{N}\}$

- $s_i = \frac{1}{n} \sum_{\ell=1}^{k(i)} n_{\ell} - \text{downward-looking status}$

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	Case 0		Case 1		Case 2		Case 3	
	$n_k$	$s'_i$	$n_k$	$s'_i$	$n_k$	$s'_i$	$n_k$	$s'_i$
<b>B</b>	0		25	1/4	0		25	1/4
<b>E</b>	50	1/2	25	1/2	50	1/2	25	1/2
<b>G</b>	25	3/4	25	3/4	50	1	50	1
<b>N</b>	25	1	25	1	0		0	
$\mu(\mathbf{s})$		11/16		5/8		3/4		11/16

- $n_k$  is # persons in category  $k \in \{\mathbf{B}, \mathbf{E}, \mathbf{G}, \mathbf{N}\}$

- $s'_i = \frac{1}{n} \sum_{\ell=k(i)}^K n_\ell$  – upward-looking status

# Four distributional scenarios (2)

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	Case 0		Case 1		Case 2		Case 3	
	$n_k$	$s_i$	$n_k$	$s_i$	$n_k$	$s_i$	$n_k$	$s_i$
<b>B</b>	0		25	1	0		25	1
<b>E</b>	50	1	25	3/4	50	1	25	3/4
<b>G</b>	25	1/2	25	1/2	50	1/2	50	1/2
<b>N</b>	25	1/4	25	1/4	0		0	
$\mu(\mathbf{s})$		11/16		5/8		3/4		11/16

- Case 0 to Case 1:
  - 25 people promoted from E to B
  - if  $e$  equals to any of values taken by  $\mu(\mathbf{s})$
  - then inequality increases

# Four distributional scenarios (3)

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	Case 0		Case 1		Case 2		Case 3	
	$n_k$	$s_i$	$n_k$	$s_i$	$n_k$	$s_i$	$n_k$	$s_i$
<b>B</b>	0		25	1	0		25	1
<b>E</b>	50	1	25	3/4	50	1	25	3/4
<b>G</b>	25	1/2	25	1/2	50	1/2	50	1/2
<b>N</b>	25	1/4	25	1/4	0		0	
$\mu(\mathbf{s})$		11/16		5/8		3/4		11/16

- Case 0 to Case 2:
  - 25 people promoted from N to G
  - if  $e$  equals to any of values taken by  $\mu(\mathbf{s})$
  - then inequality decreases

# Transfer Principle again

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	Case 0		Case 1		Case 2		Case 3	
	$n_k$	$s_i$	$n_k$	$s_i$	$n_k$	$s_i$	$n_k$	$s_i$
<b>B</b>	0		25	1	0		25	1
<b>E</b>	50	1	25	3/4	50	1	25	3/4
<b>G</b>	25	1/2	25	1/2	50	1/2	50	1/2
<b>N</b>	25	1/4	25	1/4	0		0	
$\mu(\mathbf{s})$		11/16		5/8		3/4		11/16

- Case 0 to Case 1: inequality increases
- Case 0 to Case 2: inequality decreases
- Case 0 to Case 3: combination results in ambiguous change



# Reference point

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Summary

- Inequality index requires a reference point
- **Mean status:**  $e = \eta(\mathbf{s}) = \mu(\mathbf{s})$ 
  - for continuous distributions will equal 0.5
  - for categorical data, there is no counterpart to fixed-mean assumption in income-inequality analysis
- **Median status:**  $e = \eta(\mathbf{s}) = \text{med}(\mathbf{s})$ 
  - not well-defined
  - in the example median is any value in interval  $M(\mathbf{s})$
  - $M(\mathbf{s}) = [1/2, 1)$  in cases 0 and 2
  - $M(\mathbf{s}) = [1/2, 3/4)$  in cases 1 and 3
- **Max status:**  $e = 1$ 
  - for constant  $e$  this is only value that makes sense
  - natural normalisation of index is  $c = 1$ : ensures  $I(\mathbf{1}; 1) = 0$

# Sensitivity

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- $\alpha$  captures the sensitivity of measured inequality
- If  $\alpha$  is high  $I_\alpha(\mathbf{s}; e) = \frac{1}{\alpha[\alpha-1]} \left[ \frac{1}{n} \sum_{i=1}^n \left[ \frac{s_i}{e} \right]^\alpha - c \right]$  sensitive to high status-inequality
- If  $\alpha = 0$  and  $c = 1$  then becomes  $I_0(\mathbf{s}; e) = -\frac{1}{n} \log \left( \frac{s_i}{e} \right)$
- If  $e = \mu(\mathbf{s})$  and  $\alpha = c = 1$  then we have  $I_1(\mathbf{s}; e) = \frac{1}{n} \sum_{i=1}^n \frac{s_i}{e} \log \left( \frac{s_i}{e} \right)$

# Behaviour of $I_0(\mathbf{s}; e)$

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	Case 0	Case 1	Case 2	Case 3
$\mu(\mathbf{s})$	11/16	5/8	3/4	11/16
$\text{med}_1(\mathbf{s})$	3/4	5/8	3/4	5/8
$\text{med}_2(\mathbf{s})$	1/2	1/2	1/2	1/2
$I_0(\mathbf{s}; \mu(\mathbf{s}))$	0.1451	0.1217	0.0588	0.0438
$I_0(\mathbf{s}; \text{med}_1(\mathbf{s}))$	0.2321	0.1217	0.0588	-0.0515
$I_0(\mathbf{s}; \text{med}_2(\mathbf{s}))$	-0.1732	-0.1013	-0.3465	-0.2746
$I_0(\mathbf{s}; 1)$	0.5198	0.5917	0.3465	0.4184

- $I_0(\mathbf{s}; \mu(\mathbf{s})), I_0(\mathbf{s}; \text{med}_1(\mathbf{s}))$ : inequality *decreases* when one person promoted from E to B
  - Case 0 to Case 1, or Case 2 to Case 3
  - movement changes both the  $\mu(\mathbf{s})$  and  $\text{med}_1(\mathbf{s})$  ref points
- $I_0(\mathbf{s}; \text{med}_2(\mathbf{s})) < 0$  for *all* cases in example!
- But  $I_\alpha(\mathbf{s}; 1)$  seems sensible

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# Behaviour of $I_\alpha(\mathbf{s}; 1)$ with $\alpha$

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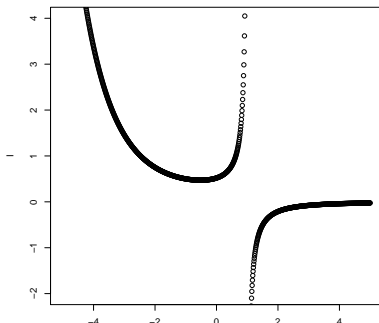
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$$\bullet I_\alpha(\mathbf{s}, 1) = \begin{cases} \frac{1}{\alpha(\alpha-1)} \left[ \frac{1}{n} \sum_{i=1}^n s_i^\alpha - 1 \right], & \text{if } \alpha \neq 0, 1, \\ -\frac{1}{n} \sum_{i=1}^n \log s_i. & \text{if } \alpha = 0. \end{cases}$$

# $I_\alpha(\mathbf{s}; 1)$ : Parameter restriction

Inequality:  
Ordinal

Cowell,  
Flachaire

- Inequality can also be written  $I_\alpha(\mathbf{s}, 1) = \frac{1}{\alpha-1} \left[ \frac{1}{n} \sum_{i=1}^n \frac{s_i^{\alpha-1}}{\alpha} \right]$

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- if  $0 < s < 1$  then  $[s^\alpha - 1]/\alpha < 0$  and if  $s = 1$  then  $[s^\alpha - 1]/\alpha = 0$
- $I_\alpha(\mathbf{s}; 1)$  only well behaved under the parameter restriction  $\alpha < 1$ .
- Alternative representation as Atkinson index on status
- $A_\alpha(\mathbf{s}) := \begin{cases} 1 - \left[ \frac{1}{n} \sum_{i=1}^n s_i^\alpha \right]^{1/\alpha} & \text{if } \alpha < 0 \text{ or } 0 < \alpha < 1, \\ 1 - \left[ \prod_{i=1}^n s_i \right]^{1/n} & \text{if } \alpha = 0. \end{cases}$

# Implementation

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- Description of sample

$$x_i = \begin{cases} 1 & \text{with sample proportion } p_1 \\ 2 & \text{with sample proportion } p_2 \\ \dots & \\ K & \text{with sample proportion } p_K \end{cases},$$

- Point estimate of the index:

$$I_\alpha = \begin{cases} \frac{1}{\alpha(\alpha-1)} \left[ \sum_{i=1}^K p_i \left[ \sum_{j=1}^i p_j \right]^\alpha - 1 \right] & \text{if } \alpha \neq 0, 1 \\ - \sum_{i=1}^K p_i \log \left[ \sum_{j=1}^i p_j \right] & \text{if } \alpha = 0 \end{cases}$$

- function of  $K$  parameter estimates  $(p_1, p_2, \dots, p_K)$  following a multinomial

# Asymptotics

Inequality:  
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- From the CLT  $I_\alpha$  is asymptotically Normally distributed

- Estimator of cov matrix of  $(p_1, p_2, \dots, p_K)$  is

$$\Sigma = \frac{1}{n} \begin{bmatrix} p_1(1-p_1) & -p_1p_2 & \dots & -p_1p_K \\ -p_2p_1 & p_2(1-p_2) & \dots & -p_2p_K \\ \vdots & \vdots & \ddots & \vdots \\ -p_Kp_1 & -p_Kp_2 & \dots & p_K(1-p_K) \end{bmatrix}$$

- $\widehat{\text{Var}}(I_\alpha) = D\Sigma D^\top$  with  $D = \left[ \frac{\partial I_\alpha}{\partial p_1} ; \frac{\partial I_\alpha}{\partial p_2} ; \dots ; \frac{\partial I_\alpha}{\partial p_K} \right]$
- $\frac{\partial I_\alpha}{\partial p_l} = \frac{1}{\alpha(\alpha-1)} \left( \left[ \sum_{i=1}^l p_i \right]^\alpha + \alpha \sum_{i=l}^{K-1} p_i \left[ \sum_{j=1}^i p_j \right]^{\alpha-1} \right), \alpha \neq 0$
- $\frac{\partial I_0}{\partial p_l} = -\log \left[ \sum_{j=1}^l p_j \right] - \sum_{i=l}^{K-1} p_i \left[ \sum_{j=1}^i p_j \right]^{-1}$

# Confidence Intervals

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- 3 variants of CIs: Asymptotic, Percentile Bootstrap, Studentized Bootstrap
- $CI_{asym} = [I_\alpha - c_{0.975} \widehat{\text{Var}}(I_\alpha)^{1/2}; I_\alpha + c_{0.975} \widehat{\text{Var}}(I_\alpha)^{1/2}]$ 
  - $c_{0.975}$  from the Student distribution  $T(n-1)$
  - do not always perform well in finite samples
- **Bootstraps**: generate resamples,  $b = 1, \dots, B$ 
  - for each resample  $b$  compute the inequality index
  - obtain  $B$  bootstrap statistics,  $I_\alpha^b$
  - also  $B$  bootstrap  $t$ -statistics  $t_\alpha^b = (I_\alpha^b - I_\alpha) / \widehat{\text{Var}}(I_\alpha^b)^{1/2}$
- $CI_{perc} = [c_{0.025}^b; c_{0.975}^b]$ 
  - $c_{0.025}^b$  and  $c_{0.975}^b$  are from EDF of bootstrap statistics
- $CI_{stud} = [I_\alpha - c_{0.975}^* \widehat{\text{Var}}(I_\alpha)^{1/2}; I_\alpha - c_{0.025}^* \widehat{\text{Var}}(I_\alpha)^{1/2}]$ 
  - $c_{0.025}^*$  and  $c_{0.975}^*$  are from EDF of the bootstrap  $t$ -statistics



# Performance Test

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- Take an example with 3 ordered categories ( $K = 3$ )
- Samples are drawn from a multinomial distribution with probabilities  $\pi = (0.3, 0.5, 0.2)$
- Is asymptotic or bootstrap distribution a good approximation of the exact distribution of the statistic?
  - if we are using 95% CIs of  $I_\alpha$
  - coverage error rate should be close to nominal rate, 0.05
- Check coverage error rate of CIs as sample size increases
  - $\alpha = -1, 0, 0.5, 0.99$
  - 199 bootstraps
  - 10 000 replications to compute error rates
  - $n = 20, 50, 100, 200, 500, 1000$

# Estimation Methods Compared

Inequality:  
Ordinal

Cowell,  
Flachaire

	$\alpha$	-1	0	0.5	0.99
Asymptotic B	$n = 20$	0.0606	0.0417	0.0598	0.0491
	$n = 500$	0.0523	0.0492	0.0521	0.0523
	$n = 1000$	0.0485	0.0540	0.0552	0.0549
Percentile B	$n = 20$	0.0384	0.0981	0.0912	0.1023
	$n = 500$	0.0509	0.0513	0.0552	0.0554
	$n = 1000$	0.0482	0.0556	0.0547	0.0551
Studentized B	$n = 20$	0.1275	0.0843	0.1041	0.1377
	$n = 500$	0.0518	0.0478	0.0429	0.0465
	$n = 1000$	0.0473	0.0522	0.0493	0.0503

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Summary

- Asymptotic CIs perform OK in finite sample
- Percentile bootstrap performs well for  $n > 50$
- Studentized bootstrap does not do well for small samples
- Reliable results for  $\alpha = 0.99$  (index is undefined for  $\alpha = 1$ )

# World values survey

Inequality:  
Ordinal

Cowell,  
Flachaire

- Life satisfaction question:

*All things considered, how satisfied are you with your life as a whole these days? Using this card on which 1 means you are “completely dissatisfied” and 10 means you are “completely satisfied” where would you put your satisfaction with your life as a whole? (code one number): Completely dissatisfied – 1 2 3 4 5 6 7 8 9 10 – Completely satisfied*

- Health question:

*All in all, how would you describe your state of health these days? Would you say it is (read out): 1 Very good, 2 Good, 3 Fair, 4 Poor.*

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# GDP and Life satisfaction

Inequality:  
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- Cross-country comparison of life satisfaction and GDP/head
  - Easterlin or happiness-income paradox
  - Weak relation internationally?
- How should we quantify life satisfaction?
  - simple linearity of Likert scale from coding?
  - exponential scale?
  - Ng (1997), Ferrer-i-Carbonell and Frijters (2004)
- Is inequality of life satisfaction related to GDP/head?
  - Use  $I_0$  and other members of the same family

# GDP and Life satisfaction (Linear)

Inequality:  
Ordinal

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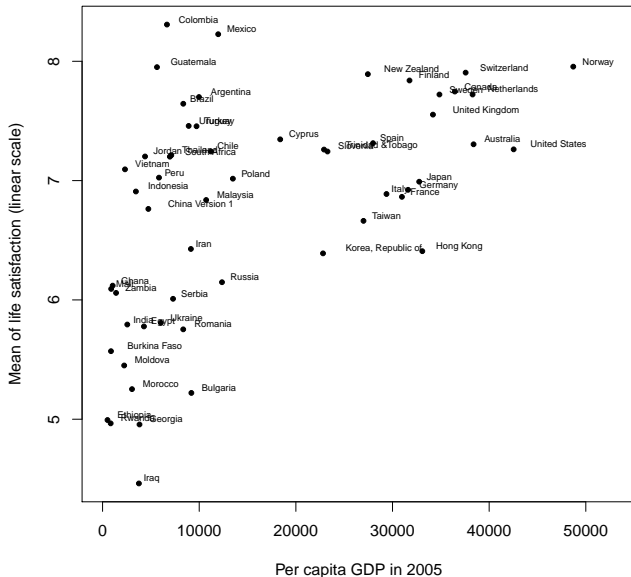
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# GDP and Life satisfaction (Exponential)

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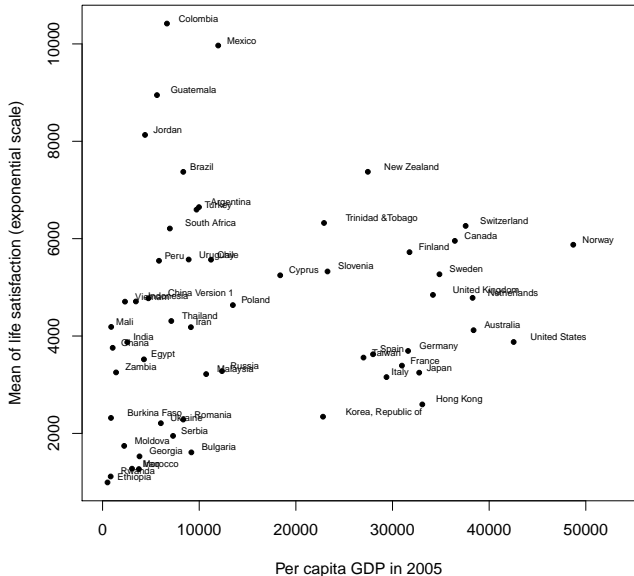
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# GDP and Inequality of Life satisfaction

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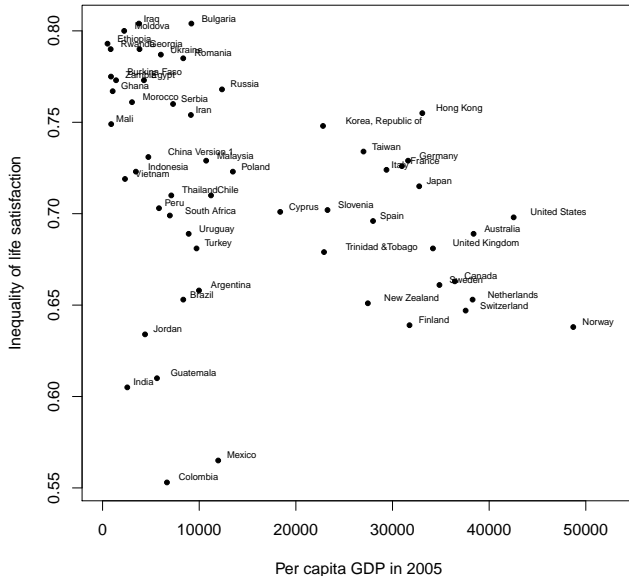
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# Health status

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Summary

- Comparison of inequality of health and the fraction of population satisfied with their health
- Cross-country comparison of inequality of health and Inequality of life satisfaction
  - use same inequality index as for life satisfaction



# Inequality of health and GDP

Inequality:  
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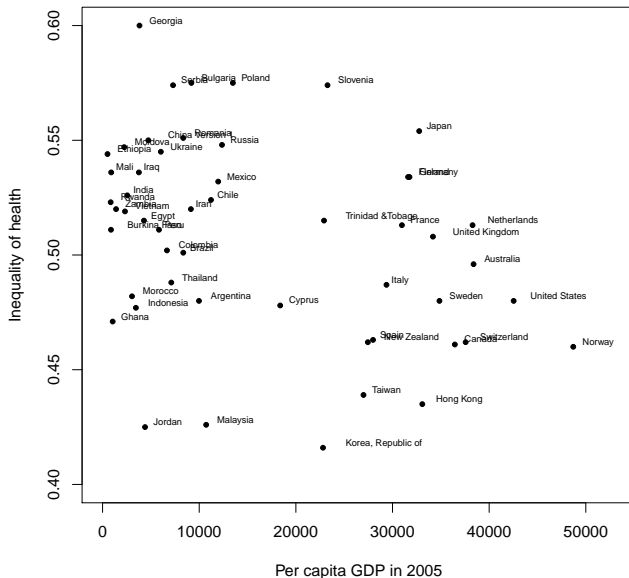
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# Inequality of health

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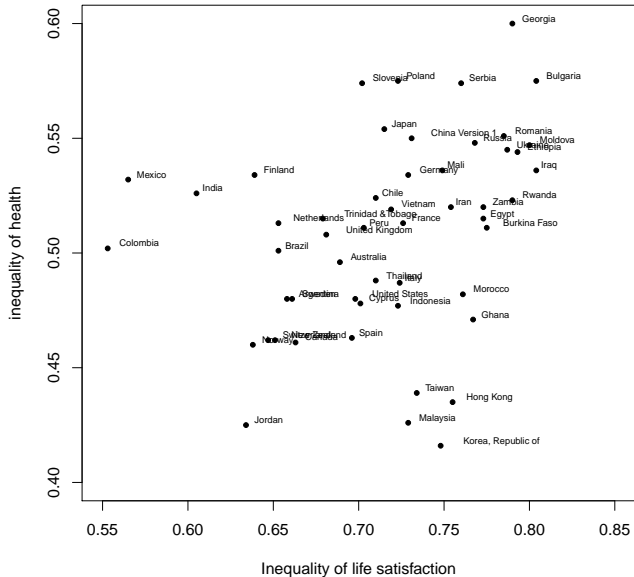
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# Application: overview

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- Satisfaction / GDP results sensitive to the cardinal interpretation of the answers
  - linear: get a positive relation below \$15 000, flat after that
  - exponential: no relation
- OLS estimate of  $I_0$ (life satisfaction) on the GDP per capita small and negative
  - happiness-income relationship is weak in cross-country comparisons
- No clear relationship between  $I_0$ (health) on GDP per capita
- OLS estimate of  $I_0$ (health) on  $I_0$ (life satisfaction) produces a slope coefficient not significantly different from zero
  - health-life satisfaction relationship is not significant

# Next Steps+

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- Theoretical tweaks
  - alternatives concepts of status
  - alternatives to scale invariance
- Interpretation in terms of inequality of opportunity
- Further empirical applications
  - Health status
  - Education

# Summary

Inequality:  
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Summary

- Inequality with ordinal data is a widespread phenomenon
- Conventional *I*-measures may make no sense
- Our approach:
  - separates out the issue of status from that of inequality-aggregation
  - allows you to choose “reference status”
  - gives a family of measures
- Nice properties empirically