GAME THEORY: DYNAMIC

MICROECONOMICS

Principles and Analysis

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Overview

Game Theory: Dynamic

- Game and subgame
- Equilibrium Issues
- Applications

Mapping the temporal structure of games
Time

- Why introduce “time” into model of a game?
  - Without it some concepts meaningless
    - can we really speak about reactions?
    - an equilibrium path?
    - threats?
- “Time” involves structuring economic decisions
  - model the sequence of decision making
  - consideration of rôle of information in that sequence
- Be careful to distinguish strategies and actions
  - see this in a simple setting
A simple game

- Stage 1: Alf’s decision
- Stage 2: Bill’s decision following [LEFT]
- Stage 2: Bill’s decision following [RIGHT]
- The payoffs
A simple game: Normal form

- **Alf** has two strategies
- **Bill** has four strategies
- The payoffs

<table>
<thead>
<tr>
<th>Alf</th>
<th>[LEFT]</th>
<th>[RIGHT]</th>
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<tbody>
<tr>
<td>[LEFT]</td>
<td>((\upsilon_1^a, \upsilon_1^b))</td>
<td>((\upsilon_1^a, \upsilon_1^b))</td>
</tr>
<tr>
<td>[RIGHT]</td>
<td>((\upsilon_3^a, \upsilon_3^b))</td>
<td>((\upsilon_4^a, \upsilon_4^b))</td>
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- **Alf moves first:** strategy set contains just two elements
- **Bill moves second:** strategy set contains four elements
The setting

- Take a game in strategic form
- If each player has exactly one round of play
  - game is extremely simple
  - simultaneous or sequential?
- Otherwise need a way of describing structure
  - imagine a particular path through the tree diagram
  - characterise unfolding decision problem
- Begin with some reminders
Structure: 1 (reminders)

- **Decision nodes**
  - in the extensive form
  - represent points where a decision is made by a player

- **Information set**
  - where is player (decision maker) located?
  - may not have observed some previous move in the game
  - knows that he is at one of a number of decision nodes
  - collection of these nodes is the information set
Structure: 2 (detail)

- **Stage**
  - a particular point in the logical time sequence of the game
  - payoffs come after the last stage

- **Direct successor nodes**
  - take the decision branches (actions) that follow from node *
  - if the branches lead to other decision nodes at the next stage
  - then these are *direct successor nodes* to node *

- **Indirect successor nodes**
  - repeat the above through at least one more stage
  - to get *indirect successor nodes*

- **How can we use this structure?**
  - break overall game down into component games?
Subgames (1)

- A *subgame* of an extensive form game
  - a subset of the game
  - satisfying three conditions

1. Begins with a “singleton” information set
   - contains a single decision node
   - just like start of overall game

2. Contains all the decision nodes that
   - are direct or indirect successors
   - and no others

3. If a decision node is in the subgame then
   - any other node in the same information set is also in the subgame
Subgames (2)

- **Stage 1:** (Alf')
- **Stage 2:** (Bill)
- **Add a stage:** (Alf again)
- **The payoffs**
- **A subgame**
- **Another subgame**
Subgames (3)

- The previous structure
- Additional strategy for Alf
- Ambiguity at stage 3
- A subgame
- Not a subgame (violates 2)
- Not a subgame (violates 3)
Game and subgame: lessons

- “Time” imposes structure on decision-making
- Representation of multistage games
  - requires care
  - distinguish actions and strategies
  - normal-form representation can be awkward
- Identifying subgames
  - three criteria
  - cases with non-singleton information sets are tricky
Overview

Game Theory: Dynamic

Game and subgame

Equilibrium Issues

Applications

Concepts and method
Equilibrium

- Equilibrium raises issues of concept and method
  - both need some care
  - as with the simple single-shot games
- Concept
  - can we use the Nash Equilibrium again?
  - clearly requires careful specification of the strategy sets
- Method
  - a simple search technique?
  - but will this always work?
- We start with an outline of the method
Backwards induction

- Suppose the game has $N$ stages
- Start with stage $N$
  - suppose there are $m$ decision nodes at stage $N$
- Pick an arbitrary node
  - suppose $h$ is player at this stage
  - determine $h$’s choice, conditional on arriving at that node
  - note payoff to $h$ and to every other player arising from this choice
- Repeat for each of the other $m - 1$ nodes
  - this completely solves stage $N$
  - gives $m$ vectors of $[v_1],..., [v_m]$
- Re-use the values from solving stage $N$
  - gives the payoffs for a game of $N - 1$ stages
- Continue on up the tree
Backwards induction: example

- Examine the last stage of the 3-stage game used earlier
- Suppose the eight payoff-levels for Alf satisfy
  - $\nu_1^a > \nu_2^a$ (first node)
  - $\nu_3^a > \nu_4^a$ (second node)
  - $\nu_5^a > \nu_6^a$ (third node)
  - $\nu_7^a > \nu_8^a$ (fourth node)
- If the game had in fact reached the first node:
  - obviously Alf would choose [LEFT]
  - so value to (Alf, Bill) of reaching first node is $[\nu_1] = (\nu_1^a, \nu_1^b)$
- Likewise the value of reaching other nodes at stage 3 is
  - $[\nu_3]$ (second node)
  - $[\nu_5]$ (third node)
  - $[\nu_7]$ (fourth node)
- Backwards induction has reduced the 3-stage game
  - to a two-stage game
Backwards induction: diagram

- 3-stage game as before
- Payoffs to 3-stage game
- Alf would play [LEFT] at this node
- and here

The 2-stage game derived from the 3-stage game
Equilibrium: questions

- Backwards induction is a powerful method
  - accords with intuition
  - usually leads to a solution
- But what is the appropriate underlying concept?
- Does it find all the relevant equilibria?
- What is the role for the Nash Equilibrium (NE) concept?
- Begin with the last of these
Equilibrium example

- The extensive form
- Bill’s choices in final stage
- Values found by backwards induction
- Alf’s choice in first stage
- The equilibrium path

- Backwards induction finds equilibrium payoff of 2 for Alf, 1 for Bill
- But what is/are the NE here?
- Look at game in normal form
### Equilibrium example: Normal form

*Alf’s two strategies*
*Bill’s four strategies*
*Payoffs*
*Best replies to $s_1^a$*
*Best reply to $s_3^b$ or to $s_4^b$*
*Best replies to $s_2^a$*
*Best reply to $s_1^b$ or to $s_2^b$*

<table>
<thead>
<tr>
<th></th>
<th>$S_1^b$ [left-left]</th>
<th>$S_2^b$ [left-right]</th>
<th>$S_3^b$ [right-left]</th>
<th>$S_4^b$ [right-right]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alf $s_1$</td>
<td>0,0</td>
<td>0,0</td>
<td>2,1</td>
<td>2,1</td>
</tr>
<tr>
<td>Alf $s_2$</td>
<td>1,2</td>
<td>1,2</td>
<td>1,2</td>
<td>1,2</td>
</tr>
</tbody>
</table>

*Nash equilibria:* $(s_2^a, s_1^b), (s_2^a, s_2^b), (s_1^a, s_3^b), (s_1^a, s_4^b)$
Equilibrium example: the set of NE

- The set of NE include the solution already found
  - backwards induction method
  - \((s_1^a, s_3^b)\) yields payoff (2,1)
  - \((s_1^a, s_4^b)\) yields payoff (2,1)

- What of the other NE?
  - \((s_2^a, s_1^b)\) yields payoff (1,2)
  - \((s_2^a, s_2^b)\) yields payoff (1,2)

- These suggest two remarks
  - First, Bill’s equilibrium strategy may induce some odd behaviour
  - Second could such an NE be sustained in practice?

- We follow up each of these in turn
Equilibrium example: odd behaviour?

- Take the Bill strategy $s_1^b = \text{[left-left]}$
  - “Play [left] whatever Alf does”
- If Alf plays [RIGHT] on Monday
  - On Tuesday it’s sensible for Bill to play [left]
- But if Alf plays [LEFT] on Monday
  - what should Bill do on Tuesday?
  - the above strategy says play [left]
  - but, from Tuesday’s perspective, it’s odd
- Given that the game reaches node *
  - Bill then does better playing [right]
  - Yet $s_1^b$ is part of a NE??
Equilibrium example: practicality

- Again consider the NE not found by backwards induction
  - give a payoff of 1 to Alf, 2 to Bill
- Could Bill “force” such a NE by a threat?
  - imagine the following pre-play conversation
  - Bill: “I will play strategy [left-left] whatever you do”
  - Alf: “Which means?”
  - Bill: “To avoid getting a payoff of 0 you had better play [RIGHT]”
- The weakness of this is obvious
  - suppose Alf goes ahead and plays [LEFT]
  - would Bill now really carry out this threat?
  - after all Bill would also suffer (gets 0 instead of 1)
- Bill’s threat seems incredible
  - so the “equilibrium” that seems to rely on it is not very impressive
Equilibrium concept

- Some NEs are odd in the dynamic context
  - so there’s a need to refine equilibrium concept

- Introduce Subgame-Perfect Nash Equilibrium (SPNE)
- A profile of strategies is a SPNE for a game if it
  - is a NE
  - induces actions consistent with NE in every subgame
NE and SPNE

- All SPNE are NE
  - reverse is not true
  - some NE that are not SPNE involve agents making threats that are not credible
- Definition of SPNE is demanding
  - it says something about all the subgames
  - even if some subgames do not interesting
  - or are unlikely to be actually reached in practice
- Backward induction method is useful
  - but not suitable for all games with richer information sets
Equilibrium issues: summary

- Backwards induction provides a practical method
- Also reveals weakness of NE concept
- Some NE may imply use of empty threats
  - given a node where a move by $h$ may damage opponent
  - but would cause serious damage $h$ himself
  - $h$ would not rationally make the move
  - threatening this move should the node be reached is unimpressive
- Discard these as candidates for equilibria?
- Focus just on those that satisfy subgame perfection
- See how these work with two applications
Overview

Industrial organisation (1)

- Game Theory: Dynamic
  - Game and subgame
  - Equilibrium Issues
  - Applications
    - Market leadership
    - Market entry
Market leadership: an output game

- Firm 1 (leader) gets to move first: chooses $q^1$
- Firm 2 (follower) observes $q^1$ and then chooses $q^2$
- Nash Equilibria?
  - given firm 2’s reaction function $\chi^2(\cdot)$
  - any $(q^1, q^2)$ satisfying $q^2 = \chi^2(q^1)$ is the outcome of a NE
  - many such NE involve incredible threats
- Find SPNE by backwards induction
  - start with follower’s choice of $q^2$ as best response to $q^1$
  - this determines reaction function $\chi^2(\cdot)$
  - given $\chi^2$ the leader's profits are $p(q^2 + \chi^2(q^1))q^1 - C^1(q^1)$
  - max this w.r.t. $q^1$ to give the solution
Market leadership

- Follower’s isoprofit curves
- Follower max profits conditional on $q^1$
- Leader’s opportunity set
- Leader’s isoprofit curves
- Leader max profits conditional on $q^1$

- Firm 2’s reaction function gives set of NE
- Stackelberg solution as SPNE
Overview

Industrial organisation (2)

- Game Theory: Dynamic
  - Game and subgame
  - Equilibrium Issues
    - Applications
      - Market leadership
      - Market entry
Entry: reusing an example

- Take the example used to illustrate equilibrium
  - recall the issue of non-credible threats
- Modify this for a model of market entry
  - rework the basic story
  - a monopolist facing possibility of another firm entering
  - will there be a fight (e.g. a price war) after entry?
  - should such a fight be threatened?
- Replace Alf with the potential entrant firm
  - [LEFT] becomes “enter the industry”
  - [RIGHT] becomes “stay out”
- Replace Bill with the incumbent firm
  - [left] becomes “fight a potential entrant”
  - [right] becomes “concede to a potential entrant”
Entry: reusing an example (more)

- **Payoffs: potential entrant firm**
  - if it enters and there’s a fight: 0
  - if stays out: 1 (profit in some alternative opportunity)
  - if enters and there’s no fight: 2

- **Payoffs: incumbent firm**
  - if it fights an entrant: 0
  - if concedes entry without a fight: 1
  - if potential entrant stays out: 2 (monopoly profit)

- **Use the equilibrium developed earlier**
  - Find the SPNE
  - We might guess that outcome depends on “strength” of the two firms
  - Let’s see
Entry example

- The original example
- The modified version
- Remove part of final stage that makes no sense
- Entrant’s choice in first stage
- Incumbent’s choice in final stage
- The equilibrium path

- SPNE is clearly (IN, concede)
- A threat of fighting would be incredible
Entry: modifying the example

- The simple result of the SPNE in this case is striking
  - but it rests on an assumption about the “strength” of the incumbent
  - suppose payoffs to the incumbent in different outcomes are altered
  - specifically, suppose that it’s relatively less costly to fight
  - what then?

- Payoffs of the potential entrant just as before
  - if it enters and there’s a fight: 0
  - if stays out: 1
  - if enters and there’s no fight: 2

- Lowest two payoffs for incumbent are interchanged
  - if it fights an entrant: 1 (maybe has an advantage on “home ground”)
  - if concedes entry without a fight: 0 (maybe dangerous to let newcomer establish a foothold)
  - if potential entrant stays out: 2 (monopoly profit)

- Take another look at the game and equilibrium
Entry example (revised)

- The example revised
- Incumbent’s choice in final stage
- Entrant’s choice in first stage

- The equilibrium path is trivial
- SPNE involves potential entrant choosing [OUT]
Entry model: development

- Approach has been inflexible
  - relative strength of the firms are just hardwired into the payoffs
  - can we get more economic insight?

- What if the rules of the game were amended?
  - could an incumbent make credible threats?

- Introduce a “commitment device”
  - example of this is where a firm incurs sunk costs
  - the firm spends on an investment with no resale value

- A simple version of the commitment idea
  - introduce an extra stage at beginning of the game
  - incumbent might carry out investment that costs $k$
  - advertising?

First, generalise the example:
Entry deterrence: two subgames

- Firm 2 chooses whether to enter
- Firm 1 chooses whether to fight
- Payoffs if there had been pre-play investment

\[
\begin{align*}
\Pi_M &: \text{monopoly profit for incumbent} \\
\Pi &: \text{reservation profit for challenger} \\
\Pi_F &: \text{incumbent’s profit if there’s a fight} \\
\Pi_J &: \text{profit for each if they split the market} \\
\text{Investment cost } k \text{ hits incumbent’s profits at each stage}
\end{align*}
\]
Entry deterrence: full model

- Firm 1 chooses whether to invest
- Firm 2 chooses whether to enter
- Firm 1 chooses whether to fight

Firm 1 chooses whether to invest
Firm 2 chooses whether to enter
Firm 1 chooses whether to fight

\[ (\Pi_F, 0) \]
\[ (\Pi_J, \Pi_J) \]
\[ (\Pi_M - k, \Pi) \]
\[ (\Pi_M, \Pi) \]
Entry deterrence: equilibrium

- Suppose the incumbent has committed to investment:
  - suppose challenger enters
  - it’s more profitable for incumbent to fight than concede if
    - \( \Pi_F > \Pi_J - k \)
- Should the incumbent precommit to investment?
  - it pays to do this rather than just allow the no-investment subgame if
    - profit from deterrence exceeds that available without investment:
      - \( \Pi_M - k > \Pi_J \)
- The SPNE is (INVEST, out) if:
  - both \( \Pi_F > \Pi_J - k \) and \( \Pi_M - k > \Pi_J \)
  - i.e. if \( k \) satisfies \( \Pi_J - \Pi_F < k < \Pi_M - \Pi_J \)
  - in this case deterrence “works”
- So it may be impossible for the incumbent to deter entry
  - in this case if \( 2\Pi_J > \Pi_M + \Pi_F \)
  - then there is no \( k \) that will work
Summary

- **New concepts**
  - Subgame
  - Subgame-perfect Nash Equilibrium
  - Backwards induction
  - Threats and credibility
  - Commitment

- **What next?**
  - extension of time idea
  - repeated games