GAMES: UNCERTAINTY

MICROECONOMICS

Principles and Analysis

Frank Cowell
Overview

Introduction to the issues

Games: Uncertainty

- Basic structure
- A model
- Illustration
Introduction

- A logical move forward in strategic analysis
  - follows naturally from considering role of time
  - similar issues arise concerning the specification of payoffs

- Important development
  - lays the basis for the economics of information
  - concerns the way in which uncertainty is treated

- Connections with analysis of risk-taking
  - make use of expected utility analysis
  - but introduce some innovative thinking.
Uncertainty

- Distinguish from simple randomisation
  - in principle it is distinct from idea of "mixed strategy"
  - but see solution method later

- Incomplete information
  - decisions made before features of the game are known
  - yet players act rationally

- Alternative perspective on uncertainty:
  - an additional player
  - "nature" makes a move

- Role of uncertainty
  - principally concerns types of player
  - may know distribution of types
  - but not the type of any one individual agent
Types

- Type concerns the nature of each agent
  - people’s tastes, behaviour may differ according to (unobserved) health status
  - firms’ costs, behaviour may differ according to (unobserved) efficiency

- Can think of type in terms of “identity”
  - same agent can take on a number of identities
  - example
    - two actual persons – Alf and Bill
    - two types – good-cop, bad-cop
    - four identities – Alf-good, Alf-bad, Bill-good, Bill-bad

- Type affects nature of strategic choices
  - which agent is which type?

- Type affects outcomes
  - payoffs that players get from individual outcomes of the game
Objectives and payoffs

- **Agents’ objectives**
  - given the “uncertainty” setting
  - makes sense to use expected utility
  - but specified how?

- **Payoffs?**
  - cardinal payoffs in each possible outcome of the game

- **Expectation?**
  - taken over joint distribution of types
  - conditional on one’s own type

- **Probabilities?**
  - determined by nature?
  - chosen by players?

- Need to look at a specific model to see how this works
Overview

Games: Uncertainty

Basic structure

A model

Illustration

Principles for incorporating uncertainty
Types

- Begin with the focus of the incomplete information: *types*
- Assume that the types of agent $h$ is a number $\tau^h \in [0,1]$
- Can cover a wide range of individual characteristics
- Example where each $h$ is either type [healthy] or type [ill]
  - let [healthy] be denoted $\tau^h = 0$
  - let [ill] be denoted $\tau^h = 1$
  - suppose $\pi$ is the probability agent $h$ is healthy
  - joint probability distribution over health types:
    $$F(\tau^h \mid [\tau]^{-h}) = \begin{cases} 
    \pi & \text{if } 0 \leq \tau^h < 1 \\
    1 & \text{if } \tau^h = 1
    \end{cases}$$
- Now consider the decision problem
Payoffs and types

- An agent’s type may affect his payoffs
  - if I become ill I may get lower level of utility from a given consumption bundle than if I stay healthy

- We need to modify the notation to allow for this

- Agent $h$’s utility is $V^h(s^h, [s]^{-h} ; \tau^h)$
  - $s^h$: $h$'s strategy
  - $[s]^{-h}$: everybody else's strategy
  - $\tau^h$: the type associated with player $h$
Beliefs, probabilities and payoffs (1)

- Agent h does not know types of the other agents
- But may have a set of beliefs about them
  - will select a strategy based on these beliefs
- Beliefs incorporated into a probabilistic model
- Represented by a distribution function $F$
  - joint probability distribution of types $\tau$ over the agents
  - assumed to be common knowledge
Beliefs, probabilities and payoffs (2)

- An example to illustrate
- Suppose Alf is revealed to be of type $\tau_0^a$
  - is about to choose [LEFT] or [RIGHT],
  - but doesn’t know Bill's type at the moment of this decision.
- Suppose there are 3 possibilities in Alf’s information set
  - $\tau_1^b, \tau_2^b, \tau_3^b$
- Alf knows the distribution of types that Bill may possess
- Can rationally assign conditional probabilities
  - conditional on type that has been realised for Alf
  - $\Pr(\tau_1^b \mid \tau_0^a), \Pr(\tau_2^b \mid \tau_0^a)$ and $\Pr(\tau_3^b \mid \tau_0^a)$
- These are Alf's beliefs about Bill’s type
Strategic problem facing Alf

- Case 1: Alf chooses L or R when Bill is type 1
- Case 2: Alf chooses L or R when Bill is type 2
- Case 3: Alf chooses L or R when Bill is type 3
- Alf’s information set
- Alf’s beliefs
Strategies again

- Recall an interpretation of pure strategies
  - like “radio buttons”
  - push one and only one of the buttons

- Now we have a more complex issue to consider
  - the appropriate strategy for $h$ may depend on $h$’s type $\tau^h$
  - so a strategy is no longer a single “button”

- Each agent’s strategy is conditioned on his type
  - strategy is a “button rule”
  - a function $\varsigma^h(\cdot)$ from set of types to set of pure strategies $S^h$
  - specifies a particular button for each possible value of the type $\tau^h$

- Example:
  - agent $h$ can be of exactly one of two types: $\tau^h \in \{[\text{healthy}], [\text{ill}]\}$
  - agent $h$’s button rule $\varsigma^h(\cdot)$ will generate one of two pure strategies $s_0^h = \varsigma^h([\text{healthy}])$ or $s_1^h = \varsigma^h([\text{ill}])$
  - according to the value of $\tau^h$ realised at the beginning of the game
Conditional strategies and utility

- Rule for agent $h$:
  - once agent $h$’s type is determined then…
  - $h$’s button rule $\zeta^h(\cdot)$…
  - generates a strategy $s^h = \zeta^h(\tau^h)$

- Likewise for all agents other than $h$
  - $[s]^{-h} = [\zeta^1(\tau^1), \ldots, \zeta^{h-1}(\tau^{h-1}), \zeta^{h+1}(\tau^{h+1}), \ldots]$?

- Agent $h$’s utility is determined by…
  - everyone’s strategies and $h$’s type: $V^h(s^h, [s]^{-h}; \tau^h)$
  - equivalently: $V^h(\zeta^1(\tau^1), \zeta^2(\tau^2), \ldots; \tau^h)$

- But others’ types unknown at time of $h$’s decision
  - Use the notation $\mathcal{E}(\cdot|\tau^h)$ to denote conditional expectation
  - expectation over $h$’s beliefs about others’ types given his own type
  - so criterion is expected utility: $\mathcal{E}(V^h(s^h, [s]^{-h}|\tau^h))$
Describing the game (1)

- **Certainty case analysed previously**
  - Objective is utility: \( \nu^h(s^h, [s]^{-h}) \)
  - Game is characterised by two objects
    - \([\nu^1, \nu^2, \ldots] \quad [S^1, S^2, \ldots] \)
      - a profile of utility functions
      - list of strategy sets

- **Game under uncertainty analysed here**
  - Objective is expected utility: \( \mathcal{E}(\nu^h(s^h, [s]^{-h} | \tau^h)) \)
  - Game is characterised by three objects
    - \([V^1, V^2, \ldots] \quad [S^1, S^2, \ldots] \quad F(\cdot) \)
      - a profile of utility functions,
      - list of strategy sets
      - joint probability distribution of types (beliefs)
Describing the game (2)

- Can recast the game in a familiar way
- Take each agent’s “button-rule” \( \varsigma^h(\cdot) \) as a redefined strategy in its own right
  - agent \( h \) gets utility \( \nu^h(\varsigma^h, [\varsigma]^{-h}) \)
  - equals \( E( V^h(s^h, [s]^{-h}| \tau^h)) \)
  - where \( \nu^h \) is as in certainty game

- Let \( S^h \) be the set of redefined strategies (“button rules”)
  - then the game \([V^1,V^2,\ldots] [S^1,S^2,\ldots] F(\cdot)\)
  - is equivalent to the game \([\nu^1,\nu^2,\ldots], [S^1,S^2,\ldots]\)

- A standard game with redefined strategy sets for each player
Equilibrium

- Re-examine meaning of equilibrium
  - a refinement
  - allows for the type of uncertainty that we have just modelled.
- Alternative representation of the game neatly introduces the idea of equilibrium
- A pure strategy Bayesian Nash equilibrium consists of
  - a profile of rules $\varsigma^*(\cdot)$
  - that is a NE of the game $[v^1, v^2, \ldots], [S^1, S^2, \ldots]$
- Means that we can just adapt standard NE definition
  - replace the ordinary strategies (“buttons”) in the NE
  - with the conditional strategies “button rules” $\varsigma^{*h}(\cdot)$ where
  - $\varsigma^{*h}(\cdot) \in \text{argmax } v^h(\varsigma^h(\cdot), [\varsigma^*(\cdot)]^{-h})$
Equilibrium: definition

- **Definition**
  - A profile of decision rules \( [\zeta^*] \) is a *Bayesian-Nash equilibrium* for the game if and only if for all \( h \) and for any \( \tau_0^h \) occurring with positive probability
    
    \[
    \mathcal{E}(V^h(\zeta^* h(\tau_0^h), [s^*]^{-h} | \tau_0^h)) \geq \mathcal{E}(V^h(s^h, [s^*]^{-h} | \tau_0^h)) \quad \text{for all } s^h \in S^h
    \]

- **Identity interpretation**
  - Bayesian equilibrium as a Nash equilibrium of a game with a larger number of players
  - if there are \( n \) players and \( m \) types
  - this setup as equivalent to a game with \( mn \) players
  - Each agent in a particular identity plays to maximise expected utility in that identity
Model: summary

- We have extended the standard analysis
  - objectives
  - strategies
  - equilibrium

- To allow for case where agents’ types are unknown
  - everything based on expected values
  - conditioned on agent’s own type

- Let’s put this to work in an example
  - illustrate equilibrium concept
  - outline a method of solution
Overview

Games: Uncertainty

Basic structure

A model

Illustration

the entry game (again)
Entry game: uncertainty

- Connected to previous lectures of strategic issues in industrial organisation
- But there’s a new twist
  - characteristics of firm 1 (the incumbent)
  - not fully known by firm 2 (an entrant)
- Firm 1 can commit to investment
  - would enhance firm 1's market position
  - might deter entry
- Cost of investment is crucial
  - firm 1 may be either high cost or low cost
  - which of these two actually applies is unknown to firm 2

Begin with a review of the certainty case
Entry and investment: certainty

- If firm 1 has not invested
- Firm 2 makes choice about entry
- Payoffs
- If firm 1 has invested
- Firm 2 makes choice about entry
- Payoffs

Now introduce uncertainty about firm 1’s costs

- If firm 2 stays out, it makes reservation profits $\Pi > 0$
- So, if firm 1 chooses [INVEST], firm 2 will choose [out]
- If firm 1 chooses [NOT INVEST], both firms get $\Pi_J$
- So, if $\Pi_M^* > \Pi_J$, firm 1 will choose [INVEST]
Entry under uncertainty: timing

- First a preliminary move by “Nature” (player 0)
  - that determines firm 1’s cost type
  - unobserved by firm 2

- Then a simultaneous moves by firms
  - firm 1, chooses whether or not to invest
  - firm 2, chooses whether or not to enter

- Analyse by breaking down problem
  - by firm 1's circumstances
  - and by behaviour

- Consider the following three cases
Entry under uncertainty: case 1

- Start with the very easy part of the model
- If Firm 1 does not invest:
  - then there is no problem about “type”
  - we’re back in the model of entry under certainty
- Then if firm 2 enters:
  - a joint-profit solution
  - both firms get payoff $\Pi_j$
- But if firm 2 stays out
  - firm 2 makes reservation profits $\Pi$
  - where $0 < \Pi < \Pi_j$
  - firm 1 makes monopoly profits $\Pi_M$
Entry under uncertainty: cases 2,3

- [2] If Firm 1 invests and is low cost:
  - Then if firm 2 enters
    - firm 1 makes profits $\Pi_J^* < \Pi_J$
    - firm 2's profits are forced to zero
  - But if firm 2 stays out
    - firm 1 gets enhanced monopoly profits $\Pi_M^* > \Pi_M$
    - firm 2 gets reservation profits $\Pi$

- [3] If Firm 1 invests and is high cost:
  - Then if firm 2 enters
    - firm 1 makes profits $\Pi_J^* - k$ (where $k > 0$)
    - firm 2's profits are forced to zero
  - But if firm 2 stays out
    - firm 1 gets enhanced monopoly profits $\Pi_M^* - k$
    - firm 2 gets reservation profits $\Pi$
Entry, investment and uncertain cost

- Game if firm 1 known as low-cost
- Game if firm 1 known as high-cost
- Preliminary stage ("nature")
- Information set, firm 2

Diagram:

0

\[ \pi^0 \] \quad \begin{array}{c} \text{[LOW]} \\ \text{[HIGH]} \end{array} \]

1

\begin{array}{c} \text{[NOT INVEST]} \\ \text{[INVEST]} \end{array}

2

\begin{array}{c} \text{[In]} \\ \text{[Out]} \end{array}

(\Pi_J, \Pi_J) \quad (\Pi_M, \Pi) \quad (\Pi_J^*, 0) \quad (\Pi_M^*, \Pi)

2

\begin{array}{c} \text{[In]} \\ \text{[Out]} \end{array}

(\Pi_J, \Pi_J) \quad (\Pi_M, \Pi) \quad (\Pi_J^* - k, 0) \quad (\Pi_M^* - k, \Pi)
The role of cost

- Outcome depends on $k$
- What is the potential advantage to firm 1 of investing?
- Assuming firm 1 is low cost:
  - if firm 2 enters: $\Pi_J^* - \Pi_J$
  - if does not enter: $\Pi_M^* - \Pi_M$
- Assuming firm 1 is high cost:
  - if firm 2 enters: $\Pi_J^* - k - \Pi_J$
  - if does not enter: $\Pi_M^* - k - \Pi_M$
- To make the model interesting assume that $k$ is large
  - $k > \max \{\Pi_J^* - \Pi_J, \Pi_M^* - \Pi_M\}$
- Then it’s never optimal for firm 1 to invest if it is high cost
- But what’s the equilibrium?
Equilibrium: methodology

- To find equilibrium, use “artificial uncertainty” as a device:
  - although we focus on pure strategies
  - it’s useful to consider a randomisation by the firms $i = 1, 2$
  - $i$ plays each of the two moves it can take with probability $(\pi^i, 1-\pi^i)$

- So define the following probabilities:
  - $\pi^0$: Pr that “Nature” endows firm 1 with low cost
  - $\pi^1$: Pr that firm 1 chooses [INVEST] given that its cost is low
  - $\pi^2$: Pr that that firm 2 chooses [In].

- Nature of the following probabilities:
  - $\pi^0$: exogenous and common knowledge.
  - $\pi^1, \pi^2$: chosen optimally firms

- A pure-strategy equilibrium is one where
  - $\pi^1$ is either 0 or 1 and
  - $\pi^2$ is either 0 or 1
Firm 1’s expected profits

- Consider expected $\Pi^1$ conditional on investment decision:
  - if 1 does not invest: $K := \pi^2 \Pi_J + [1 - \pi^2] \Pi_M$
  - if 1 invests and is low cost: $K^* := \pi^2 \Pi_J^* + [1 - \pi^2] \Pi_M^*$
  - if 1 invests and is high cost: [not relevant, by assumption]

- Therefore we compute expected profits as
  - $E\Pi^1 = \pi^0[\pi^1K^* + [1 - \pi^1]K] + [1 - \pi^0]K$

- Simplifying this we get
  - $E\Pi^1 = K + [K^* - K] \pi^0\pi^1$

- So expected profits increase with $\pi^1$ if $K^* > K$
  - this condition is equivalent to requiring $\pi^2 < 1 / [1 + \gamma]$
  - where $\gamma := [\Pi_J^* - \Pi_J] / [\Pi_M^* - \Pi_M]$

- Likewise expected profits decrease with $\pi^1$ if $K^* > K$
  - the case where $\pi^2 > 1 / [1 + \gamma]$
Firm 2’s expected profits

- Consider expected $\Pi^2$ conditional on investment decision:
  - if 1 does not invest: $H := \pi^2 \Pi_J + [1 - \pi^2] \Pi$
  - if 1 invests and is low cost: $H^* := [1 - \pi^2] \Pi$
  - if 1 invests and is high cost: [not relevant, by assumption]

- Therefore we compute expected profits as
  - $E \Pi^2 = \pi^0 [\pi^1 H^* + [1 - \pi^1] H] + [1 - \pi^0] H$

- Simplifying this we get
  - $E \Pi^2 = H + [H^* - H] \pi^0 \pi^1$
  - $E \Pi^2 = \Pi + \pi^2 [\Pi_J - \Pi - \pi^0 \pi^1 \Pi_J]$

- So expected profits increase with $\pi^2$ if
  - $\pi^1 < \frac{\Pi_J - \Pi}{\pi^0 \Pi_J}$
An equilibrium

- Work back from the last stage

- Firm 2’s decision:
  - increase $\pi_2$ up to its max value ($\pi_2 = 1$)
  - as long as $\pi_1 < [\Pi_j - \Pi] / [\pi^0 \Pi_j]$
  - if firm 1 has set $\pi_1 = 0$ then clearly this condition holds

- Firm 1’s decision:
  - decrease $\pi_1$ to its min value ($\pi_1 = 0$)
  - as long as $\pi_2 > 1 / [1 + \gamma]$
  - this condition obviously holds if $\pi_2 = 1$ in the final stage

- So there is a NE such that
  - “$\pi_2 = 1$” is the best response to “$\pi_1 = 0$”
  - “$\pi_1 = 0$” is the best response to “$\pi_2 = 1$”

- In this NE
  - $\pi_1 = 0$ means firm 1 chooses [NOT INVEST]
  - $\pi_2 = 1$ means firm 2 chooses [In]
Another equilibrium?

- Again work back from the last stage
- Firm 2’s decision:
  - decrease $\pi^2$ up to its min value ($\pi^2 = 0$)
  - as long as $\pi^1 > \left[ \Pi - \Pi \right] / \left[ \pi^0 \Pi \right]$
  - this condition can only hold if $\pi^0$ is “large enough”: $\pi^0 \geq 1 - \Pi / \Pi$
- Firm 1’s decision:
  - increase $\pi^1$ up to its max value ($\pi^1 = 1$)
  - as long as $\pi^2 < 1 / \left[ 1 + \gamma \right]$
  - this condition obviously holds if $\pi^2 = 0$ in the final stage
- So if $\pi^0$ is large enough there is a NE such that
  - “$\pi^2 = 0$” is the best response to “$\pi^1 = 1$”
  - “$\pi^1 = 1$” is the best response to “$\pi^2 = 0$”
- In this NE
  - $\pi^1 = 1$ means firm 1 chooses [INVEST]
  - $\pi^2 = 0$ means firm 2 chooses [Out]
The entry game: summary

- Method similar to many simple games
  - simultaneous moves
  - find mixed-strategy equilibrium

- But there may be multiple equilibria

- We find one or two NEs in pure strategies
  - [NOT INVEST] [In] – always an equilibrium
  - [INVEST] [Out] – equilibrium if firm1 is likely to be low-cost

- There may also be a mixed-strategy
  - if firm1 is likely to be low-cost
Summary

- New concepts
  - Nature as a player
  - Bayesian-Nash equilibrium

- Method
  - Visualise agents of different types as though they were different agents
  - Extend computation of NE
  - To maximisation of expected payoff

- What next?
  - Economics of Information
  - See presentation on Adverse selection