Almost essential
Firm: Optimisation
Consumption: Basics

CONSUMER OPTIMISATION

MICROECONOMICS
Principles and Analysis
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What we’re going to do:

- We’ll solve the consumer's optimisation problem
  - using methods that we've already introduced
- This enables us to re-cycle old techniques and results
- A tip:
  - check the presentation for firm optimisation
  - look for the points of comparison
  - try to find as many reinterpretations as possible
The problem

- Maximise consumer’s utility \( U(x) \)
  - \( U \) assumed to satisfy the standard “shape” axioms

- Subject to feasibility constraint \( x \in X \)
  - Assume consumption set \( X \) is the non-negative orthant

- and to the budget constraint
  \[
  \sum_{i=1}^{n} p_i x_i \leq y
  \]
  - The version with fixed money income
Two fundamental views of consumer optimisation
An obvious approach?

- We have the elements of a standard constrained optimisation problem:
  - the constraints on the consumer
  - the objective function

- The next steps might seem obvious:
  - set up a standard Lagrangian
  - solve it
  - interpret the solution

- But the obvious approach is not always the most useful
  - we will use a round-about approach
  - generates extra insights
  - enables connection with theory of the firm
Think laterally

- In microeconomics an optimisation problem can often be represented in more than one form.
- Which form you use depends on:
  - your interpretation of the problem
  - the information you want to get from the solution
- This applies here.
- The same consumer optimisation problem can be seen in two different ways:
  1. “primal problem”
  2. “dual problem”
     - standard labels in the literature
A five-point plan

1. Set out the basic consumer optimisation problem
   - the *primal* problem

2. Show that the solution is equivalent to another problem
   - the *dual* problem

3. Show that this problem is identical to that of the firm

4. Write down the solution
   - copy directly from the solution to the firm’s problem

5. Go back to the problem we first thought of
   - the primal problem again
The primal problem

The consumer aims to maximise utility...

Subject to budget constraint

Defines the primal problem

Solution to primal problem

max $U(x)$ subject to

$$\sum_{i=1}^{n} p_i x_i \leq y$$

There's another way of looking at this
The dual problem

- Alternatively the consumer could aim to minimise cost...
- Subject to utility constraint
- Defines the dual problem
- Solution to the problem
- Cost minimisation by the firm

minimise \[ \sum_{i=1}^{n} p_i x_i \]

subject to \( U(x) \geq \upsilon \)

- But where have we seen the dual problem before?
Two types of cost minimisation

- The similarity between the two problems is not just a curiosity
- We can use it to save ourselves work
- All the results that we had for the firm's “stage 1” problem can be used
- We just need to “translate” them intelligently
  - Swap over the symbols
  - Swap over the terminology
  - Relabel the theorems
Overview

Reusing results on optimisation
A lesson from the firm

- Compare cost-minimisation for the firm...
- ...and for the consumer

- The difference is only in notation
- So their solution functions and response functions must be the same

Run through formal stuff
Cost-minimisation: strictly quasiconcave $U$

- Minimise

$$\sum_{i=1}^{n} p_i x_i + \lambda [\upsilon - U(x)]$$

- Because of strict quasiconcavity we have an interior solution

- A set of $n + 1$ First-Order Conditions

$$\begin{align*}
\lambda^* U_1(x^*) &= p_1 \\
\lambda^* U_2(x^*) &= p_2 \\
\cdots & \cdots \\
\lambda^* U_n(x^*) &= p_n \\
\upsilon &= U(x^*)
\end{align*}$$

- Use the objective function
- ...and utility constraint
- ...to build the Lagrangian
- Differentiate w.r.t. $x_1, \ldots, x_n$ and set equal to 0
- ... and w.r.t. $\lambda$
- Denote cost minimising values with a $*$
If ICs can touch the axes…

- Minimise
  \[
  \sum_{i=1}^{n} p_i x_i + \lambda [\nu - U(x)]
  \]

- Now there is the possibility of corner solutions

- A set of \( n + 1 \) First-Order Conditions

\[
\begin{align*}
\lambda^* U_1(x^*) & \leq p_1 \\
\lambda^* U_2(x^*) & \leq p_2 \\
... & ... \\
\lambda^* U_n(x^*) & \leq p_n \\
\nu = U(x^*)
\end{align*}
\]

Can get “<” if optimal value of this good is 0
From the FOC

- If both goods $i$ and $j$ are purchased and MRS is defined then...

\[
\frac{U_i(x^*)}{U_j(x^*)} = \frac{p_i}{p_j}
\]

- MRS = price ratio

- If good $i$ could be zero then...

\[
\frac{U_i(x^*)}{U_j(x^*)} \leq \frac{p_i}{p_j}
\]

- MRS$_{ji}$ ≤ price ratio

- “implicit” price = market price

- “implicit” price ≤ market price

Solution
The solution...

- Solving the FOC, get a cost-minimising value for each good...

  \[ x_i^* = H^i(p, \upsilon) \]

- …for the Lagrange multiplier

  \[ \lambda^* = \lambda^*(p, \upsilon) \]

- …and for the minimised value of cost itself

- The consumer’s cost function or expenditure function is defined as

  \[ C(p, \upsilon) := \min \sum p_i x_i \quad \{U(x) \geq \upsilon\} \]
The cost function has the same properties as for the firm

- Non-decreasing in every price, increasing in at least one price
- Increasing in utility $\nu$
- Concave in $p$
- Homogeneous of degree 1 in all prices $p$
- Shephard's lemma
Other results follow

- Shephard's Lemma gives demand as a function of prices and utility:

\[ H^i(p, \nu) = C_i(p, \nu) \]

- Properties of the solution function determine behaviour of response functions.

- “Short-run” results can be used to model side constraints.

*H* is the “compensated” or conditional demand function. Downward-sloping with respect to its own price, etc… For example rationing.
Comparing firm and consumer

- Cost-minimisation by the firm…
- …and expenditure-minimisation by the consumer
- …are effectively identical problems
- So the solution and response functions are the same:

**Firm**

- Problem: \( \min_z \sum_{i=1}^{m} w_i z_i + \lambda [q - \phi(z)] \)
- Solution function: \( C(w, q) \)
- Response function: \( z_i^* = H_i(w, q) \)

**Consumer**

- Problem: \( \min_x \sum_{i=1}^{n} p_i x_i + \lambda [\nu - U(x)] \)
- Solution function: \( C(p, \nu) \)
- Response function: \( x_i^* = H_i(p, \nu) \)
Overview

Consumer: Optimisation

- Primal and Dual problems
- Lessons from the Firm

- Exploiting the two approaches

Primal and Dual again
The Primal and the Dual…

- There’s an attractive symmetry about the two approaches to the problem
- In both cases the $p$s are given and you choose the $x$s
- But constraint in the primal becomes objective in the dual…
- …and vice versa
A neat connection

- Compare the primal problem of the consumer...
- ...with the dual problem

- Two aspects of the same problem
- So we can link up their solution functions and response functions

Run through the primal
Utility maximisation

- Maximise
  \[ U(x) + \mu \left[ y - \sum_{i=1}^{n} p_i x_i \right] \]

- If \( U \) is strictly quasiconcave we have an interior solution

- A set of \( n+1 \) First-Order Conditions
  \[
  \begin{align*}
  U_1(x^*) &= \mu^* p_1 \\
  U_2(x^*) &= \mu^* p_2 \\
  \vdots & \quad \vdots \\
  U_n(x^*) &= \mu^* p_n \\
  y &= \sum_{i=1}^{n} p_i x_i^* 
  \end{align*}
  \]

- Use the objective function
- …and budget constraint
- …to build the Lagrangian
- Differentiate w.r.t. \( x_1, \ldots, x_n \) and set equal to 0
- … and w.r.t \( \mu \)
- Denote utility maximising values with a *

- Interpretation

If \( U \) not strictly quasiconcave then replace “=” by “≤”
From the FOC

- If both goods \( i \) and \( j \) are purchased and MRS is defined then...

\[
\frac{U_i(x^*)}{U_j(x^*)} = \frac{p_i}{p_j}
\]

- \( MRS = \text{price ratio} \)

- If good \( i \) could be zero then...

\[
\frac{U_i(x^*)}{U_j(x^*)} \leq \frac{p_i}{p_j}
\]

- \( MRS_{ji} \leq \text{price ratio} \)

-same as before

- “implicit” price \( = \) market price

- “implicit” price \( \leq \) market price

Solution
The solution…

- Solving the FOC, you get a utility-maximising value for each good…

\[ x^*_i = D^i(p, y) \]

- …for the Lagrange multiplier

\[ \mu^* = \mu^*(p, y) \]

- …and for the maximised value of utility itself

- The indirect utility function is defined as

\[ V(p, y) := \max \ U(x) \]

\[ \{ \sum p_i x_i \leq y \} \]
A useful connection

- The indirect utility function maps prices and budget into max utility
  \[ \nu = V(p, y) \]

- The cost function maps prices and utility into min budget
  \[ y = C(p, \nu) \]

- Therefore we have:
  \[ \nu = V(p, C(p, \nu)) \]
  \[ y = C(p, V(p, y)) \]

The indirect utility function works like an "inverse" to the cost function.
The two solution functions have to be consistent with each other. Two sides of the same coin.
Odd-looking identities like these can be useful.
The Indirect Utility Function has some familiar properties…

(All of these can be established using the known properties of the cost function)

- Non-increasing in every price, decreasing in at least one price
- Increasing in income $y$
- Quasi-convex in prices $p$
- Homogeneous of degree zero in $(p, y)$
- Roy's Identity
Roy's Identity

\[ \nu = V(p, y) = V(p, C(p, \nu)) \]

0 = \( V_i(p, C(p, \nu)) + V_y(p, C(p, \nu)) C_i(p, \nu) \)

0 = \( V_i(p, y) + V_y(p, y) x_i^* \)

\( x_i^* = -\frac{V_i(p, y)}{V_y(p, y)} \)

Marginal disutility of price \( i \)
Marginal utility of money income

Use the definition of the optimum
Differentiate w.r.t. \( p_i \)
Use Shephard's Lemma
Rearrange to get...
So we also have...

\[ x_i^* = -\frac{V_i(p, y)}{V_y(p, y)} = D_i(p, y) \]

Ordinary demand function
Utility and expenditure

- Utility maximisation
- …and expenditure-minimisation by the consumer
- …are effectively two aspects of the same problem
- So their solution and response functions are closely connected:

\[
\begin{align*}
\textbf{Primal} & \quad \textbf{Dual} \\
\text{Problem:} \quad & \max_x U(x) + \mu \left[ y - \sum_{i=1}^{n} p_i x_i \right] \\
& \min_x \sum_{i=1}^{n} p_i x_i + \lambda \left[ \nu - U(x) \right] \\
\text{Solution function:} \quad & V(p, y) \\
& C(p, \nu) \\
\text{Response function:} \quad & x_i^* = D^i(p, y) \\
& x_i^* = H^i(p, \nu)
\end{align*}
\]
Summary

- A lot of the basic results of the consumer theory can be found without too much hard work.

- We need two “tricks”:

  1. A simple relabelling exercise:
     - cost minimisation is reinterpreted from output targets to utility targets

  2. The primal-dual insight:
     - utility maximisation subject to budget is equivalent to cost minimisation subject to utility
1. Cost minimisation: two applications

- **THE FIRM**
  - min cost of inputs
  - subject to output target
  - Solution is of the form $C(w, q)$

- **THE CONSUMER**
  - min budget
  - subject to utility target
  - Solution is of the form $C(p, \nu)$
2. Consumer: equivalent approaches

- **PRIMAL**
  - max utility
  - subject to budget constraint
  - Solution is a function of \((p, y)\)

- **DUAL**
  - min \(budget\)
  - subject to utility constraint
  - Solution is a function of \((p, v)\)
Basic functional relations

- \( C(p, \nu) \) \quad \text{cost (expenditure)}
- \( H^i(p, \nu) \) \quad \text{compensated demand for good } i
- \( V(p, y) \) \quad \text{indirect utility}
- \( D^i(p, y) \) \quad \text{ordinary demand for input } i

Utility

\( H \) is also known as "Hicksian" demand

money income
What next?

- Examine the response of consumer demand to changes in prices and incomes
- Household supply of goods to the market
- Develop the concept of consumer welfare