Distributional Analysis and Inequality

HMRC-HMT Economics of Taxation
http://darp.lse.ac.uk/HMRC-HMT

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Overview...

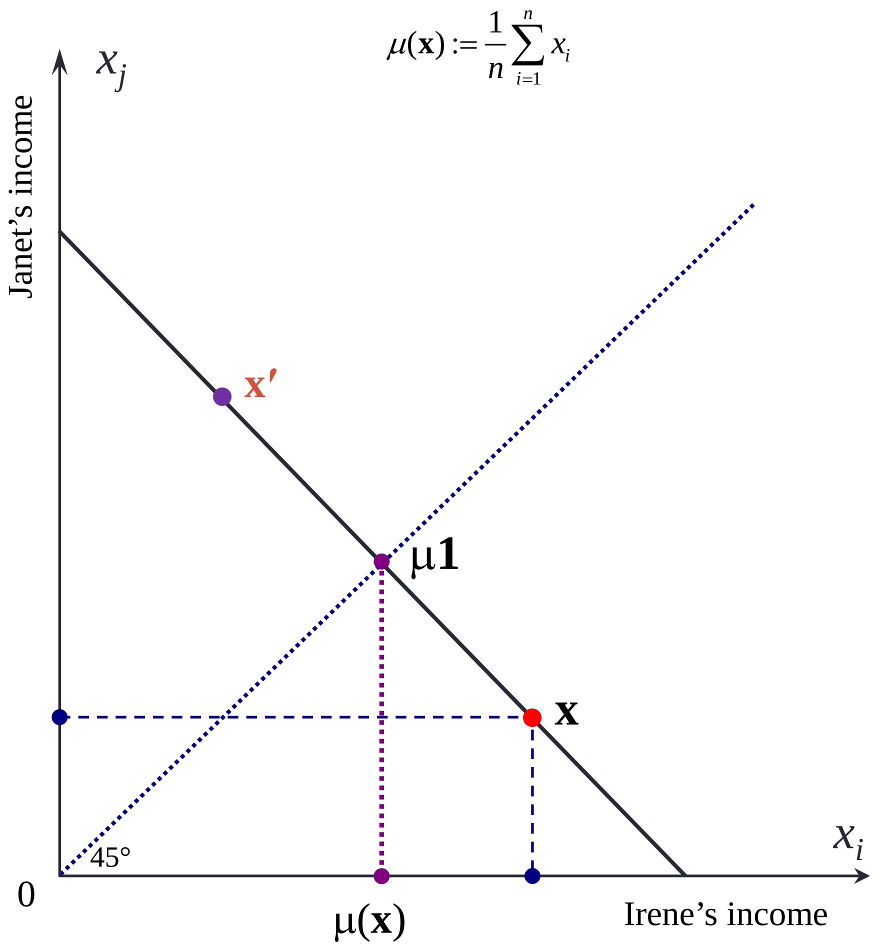
- Distributional analysis & inequality
  - **Inequality basics**
  - Social welfare
  - Ranking
  - Evidence

Figuring out inequality from first principles

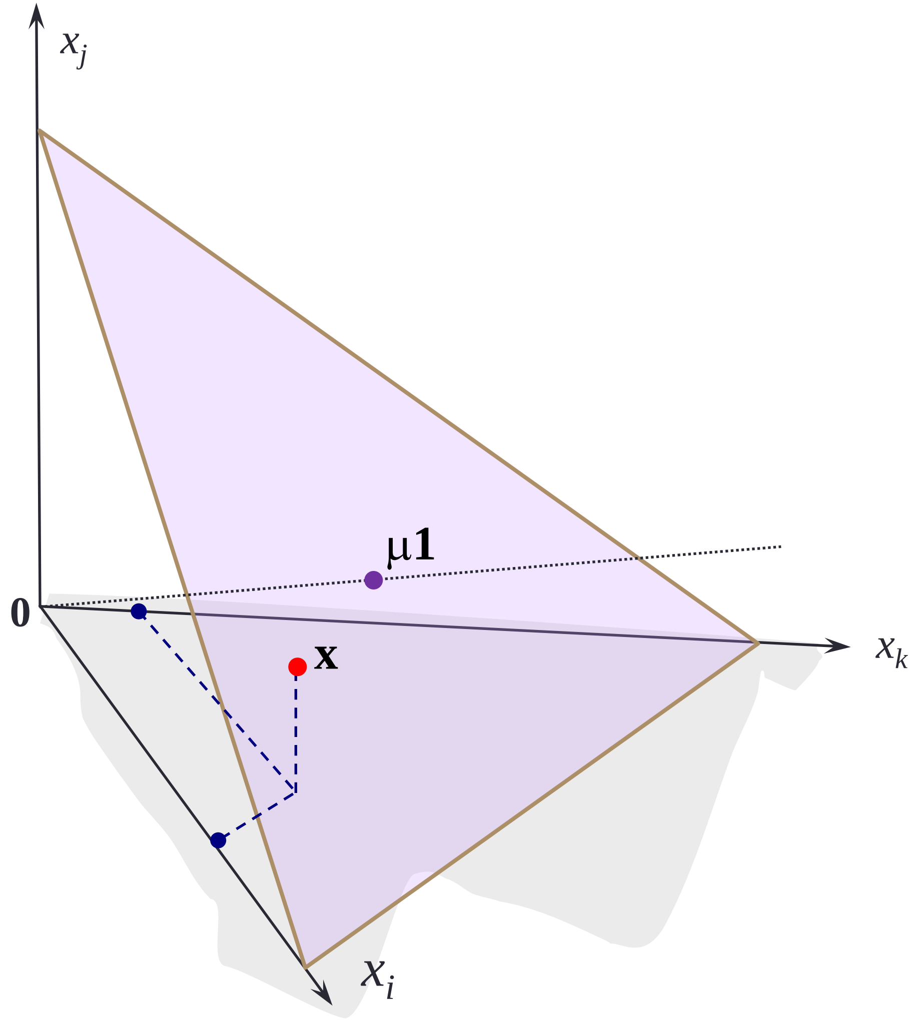
Distributional analysis

- Covers a broad class of economic problems
  - inequality
  - social welfare
  - poverty
- Similar techniques
  - rankings
  - measures
- Four basic components need to be clarified
  - "income" concept…
  - "income receiving unit" concept
  - a distribution
  - method of assessment or comparison
- See Cowell (2000, 2008, 2011, 2016), Sen and Foster (1997)

Income distributions $n = 2$



Income distributions $n = 3$



- *A representation with 3 incomes*
- *Income distributions with given total*
- *Equal income distributions*
- *income distribution* x

A fundamental question

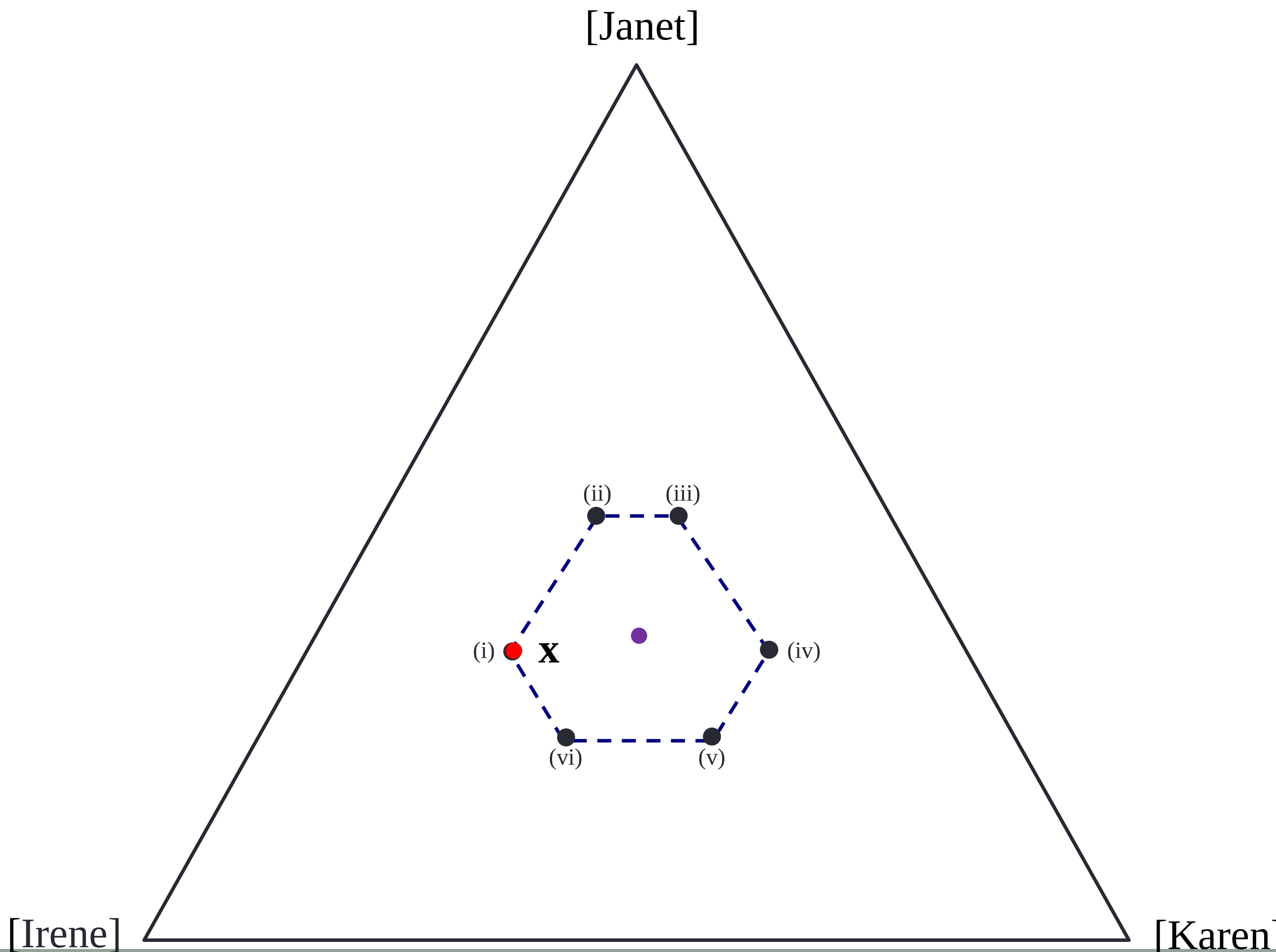
- What makes a "good" set of principles?
- There is no such thing as a "right" or "wrong" axiom.
- However axioms could be appropriate or inappropriate
  - Need some standard of "reasonableness"
  - For example, how do people view income distribution comparisons?
- Use a simple framework to list some of the basic axioms
  - Assume a fixed population of size *n*.
  - Assume that individual utility can be measured by *x*
  - Income normalised by equivalence scales
- Follow the approach of Amiel-Cowell (1999) Appendix A

Inequality axioms (1)

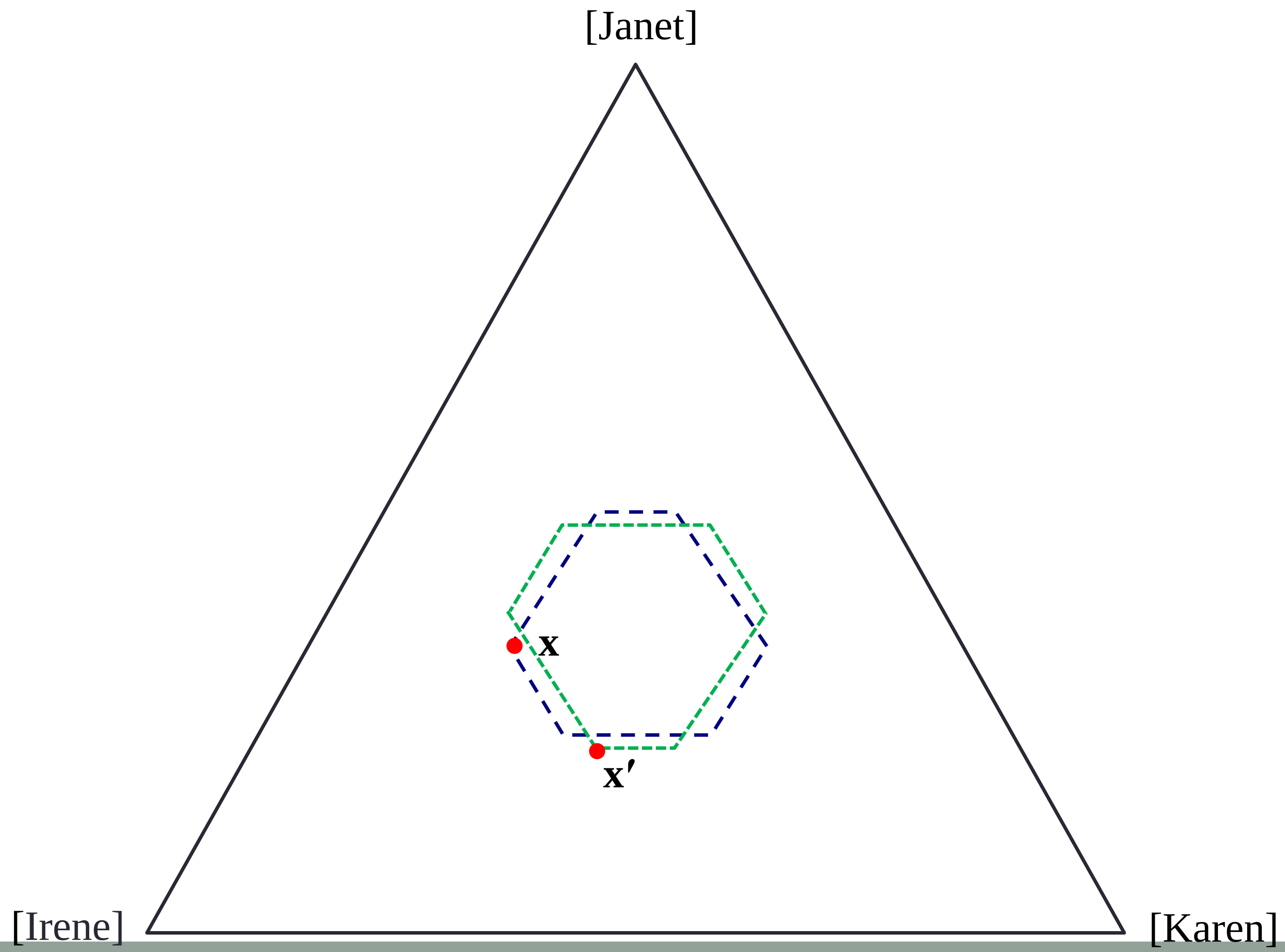
- **1 Anonymity.** Suppose $\mathbf{x}'$ is a permutation of $\mathbf{x}$. Then:
  $I(\mathbf{x}') = I(\mathbf{x})$
- **2 Population principle.**
  $I(\mathbf{x}) \geq I(\mathbf{y}) \Rightarrow I(\mathbf{x},\mathbf{x},\ldots,\mathbf{x}) \geq I(\mathbf{y},\mathbf{y},\ldots,\mathbf{y})$
- **3 Transfer principle.** (Dalton 1920) Suppose $x_i < x_j$ then, for small δ**:**

$$I(x_1,x_2\ldots, x_i + \delta,\ldots, x_j - \delta,\ldots, x_n) < I(x_1,x_2,\ldots, x_i,\ldots, x_n)$$

Income distributions $n = 3$ (close-up)



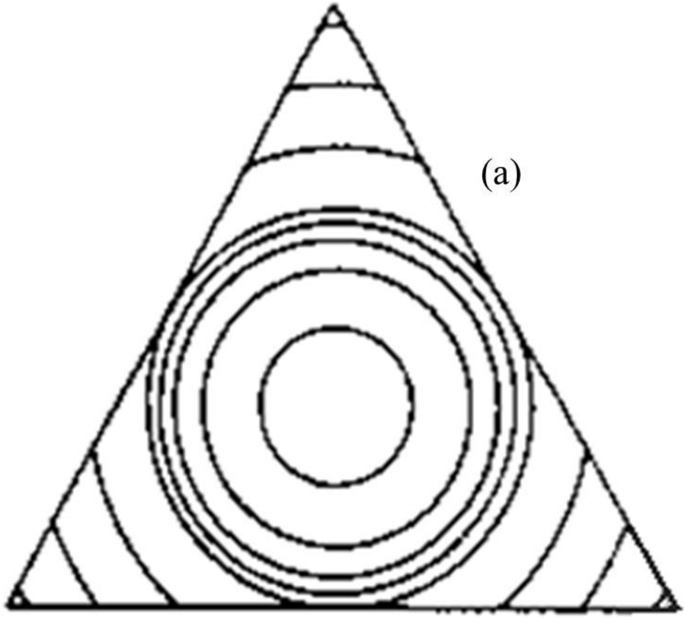
x and **x′** cannot be ranked



Two contour maps

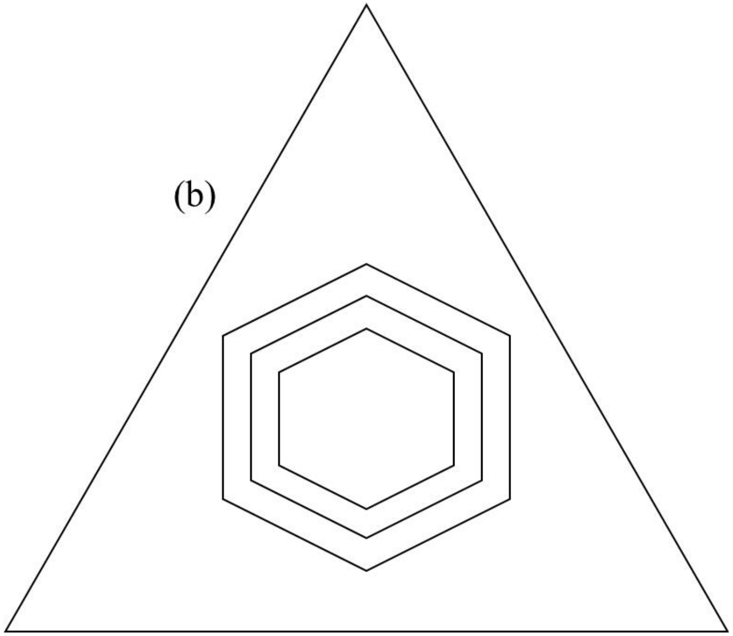
$$I_{\mathrm{CV}}(\mathbf{x}) := \frac{\sqrt{\mathrm{var}(\mathbf{x})}}{\mu(\mathbf{x})}.$$

$$I_{\mathrm{Gini}}(\mathbf{x}) := \frac{1}{2n^2\mu(\mathbf{x})}\sum_{i=1}^{n}\sum_{j=1}^{n}|\, x_i - x_j \,|.$$

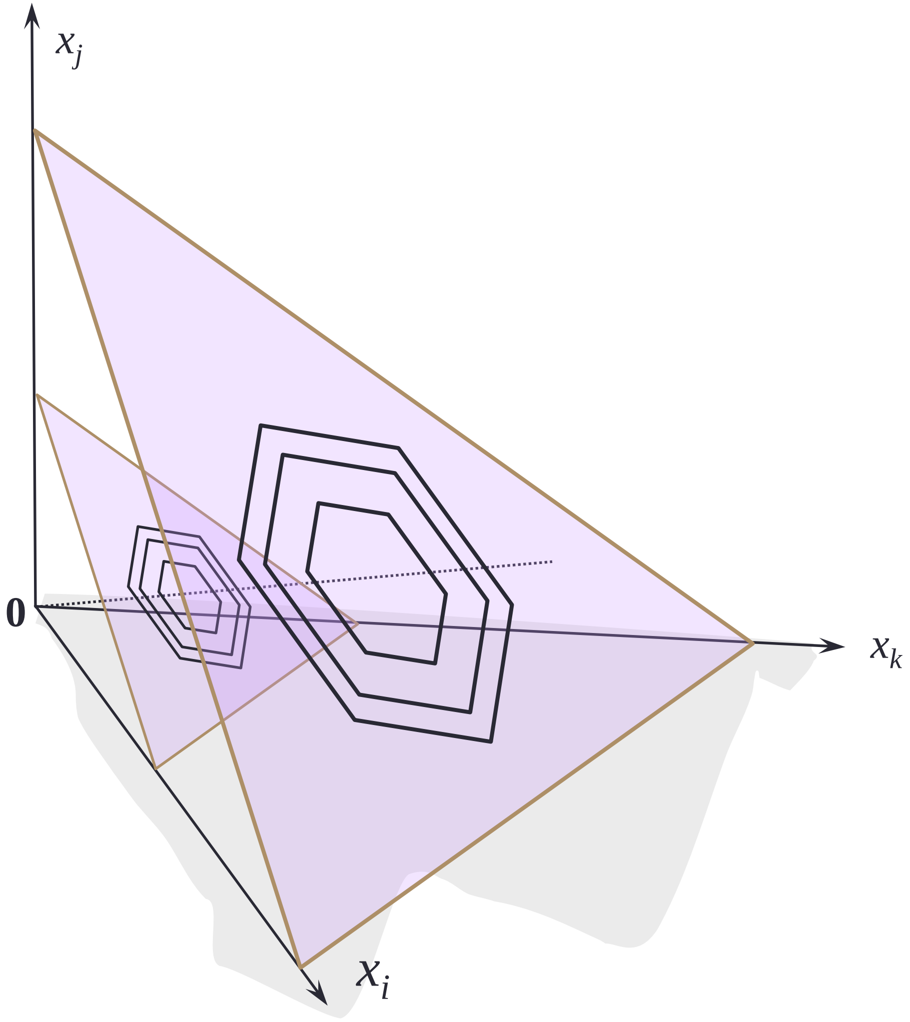


(a)

(b)



Scale invariance



Inequality axioms (2)

- **4 Decomposability**. Suppose **x**' is formed by joining **x** with **z** and **y**' is formed by joining **y** with **z**. Then **:**

  $I(\mathbf{x}) \geq I(\mathbf{y}) \Rightarrow I(\mathbf{x}') \geq I(\mathbf{y}')$
- **5 Scale invariance.** For $\lambda > 0$: $I(\mathbf{x}) \geq I(\mathbf{y}) \Rightarrow I(\lambda\mathbf{x}) \geq I(\lambda\mathbf{y})$
- **6 Translation invariance.** $I(\mathbf{x}) \geq I(\mathbf{y}) \Rightarrow I(\mathbf{x}+\mathbf{1}\delta) \geq I(\mathbf{y}+\mathbf{1}\delta)$

- Axioms 1-5 yield the Generalised Entropy class of indices

$$I_{\mathrm{GE}}^{\alpha}(\mathbf{x}) = \frac{1}{\alpha^2-\alpha}\left[\frac{1}{n}\sum_{i=1}^{n}\left[\frac{x_i}{\mu(\mathbf{x})}\right]^{\alpha} - 1\right]$$

- Axioms 1-4 + 6 yield the Kolm class + variance

$$I_{\mathrm{K}}^{\beta}(\mathbf{x}) := \frac{1}{\beta}\log\left(\frac{1}{n}\sum_{i=1}^{n} e^{\beta[x_i-\mu(\mathbf{x})]}\right)$$

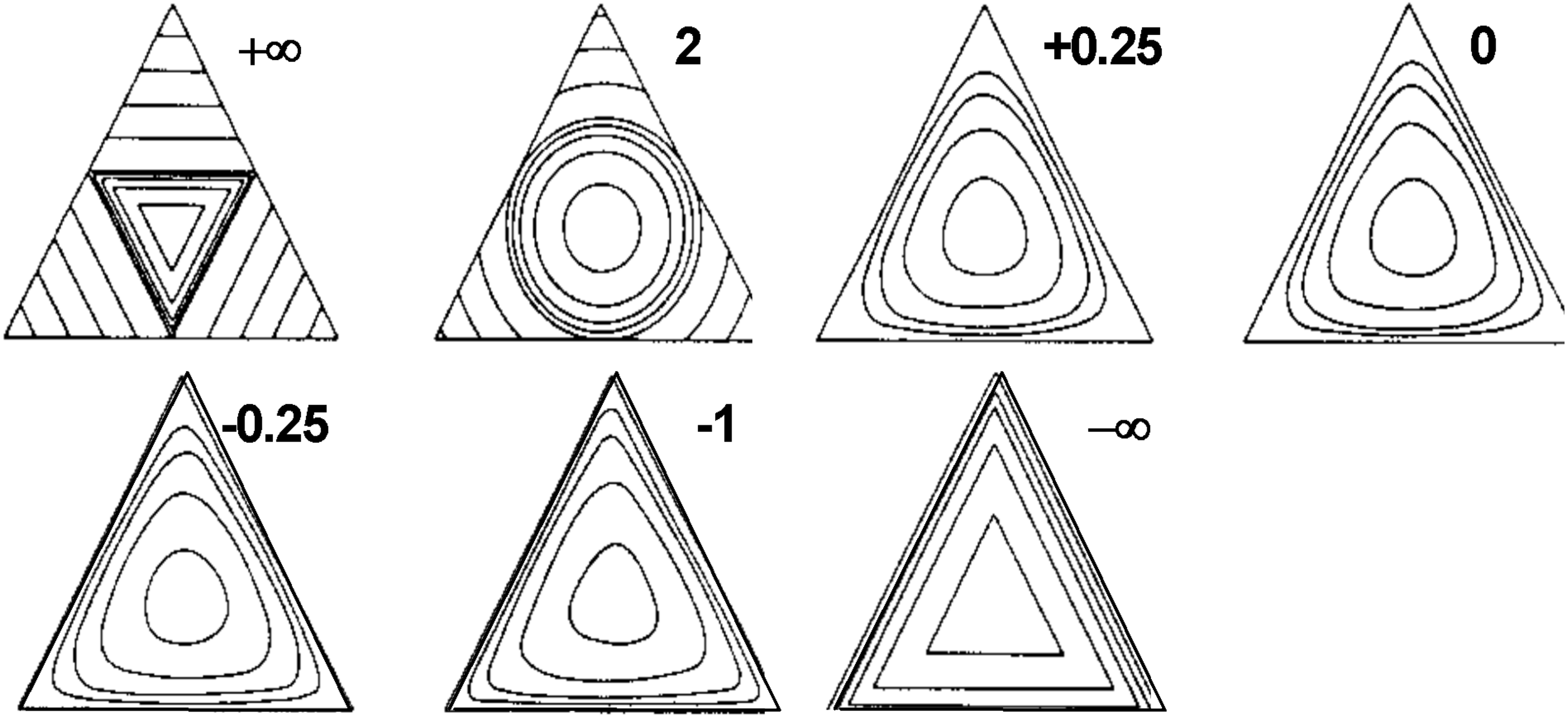
Generalised Entropy measures

- Defines a *class* of inequality measures, given parameter α :

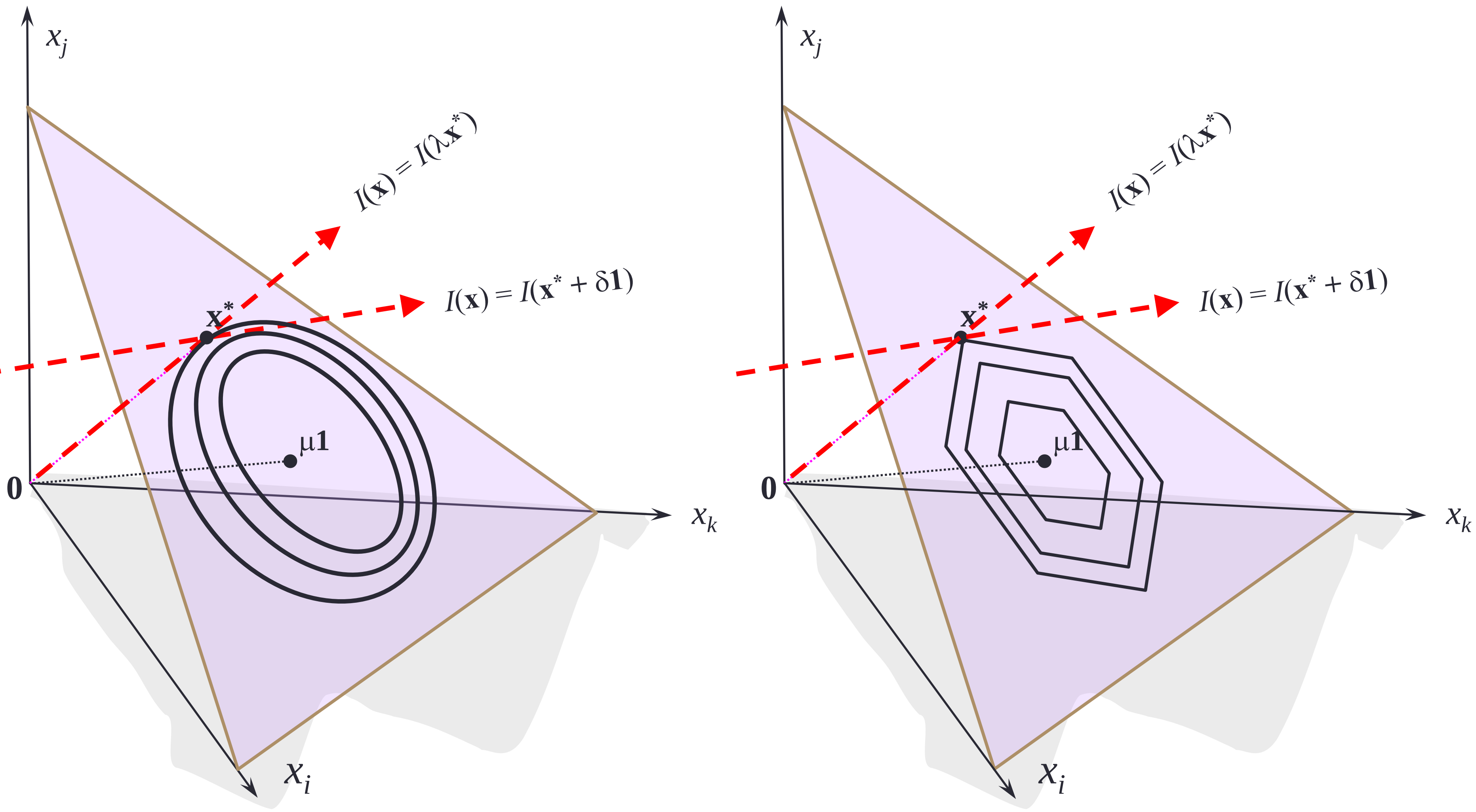
$$I_{\mathrm{GE}}^{\alpha}(\mathbf{x}) = \frac{1}{\alpha^2 - \alpha}\left[\frac{1}{n}\sum_{i=1}^{n}\left[\frac{x_i}{\mu(\mathbf{x})}\right]^{\alpha} - 1\right]$$

- GE class is rich. Some important special cases
  - for α < 1 it is ordinally equivalent to Atkinson (α = 1 – ε)
  - α = 0: $I_{\mathrm{GE}}^{0}(\mathbf{x}) := -\frac{1}{n}\sum_{i=1}^{n}\log(x_i/\mu(\mathbf{x}))$ (mean logarithmic deviation)
  - α = 1: $I_{\mathrm{GE}}^{1}(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n}[x_i/\mu(\mathbf{x})]\log(x_i/\mu(\mathbf{x}))$ (the Theil index)
  - or α = 2 it is ordinally equivalent to (normalised) variance.
- Parameter α can be assigned any positive or negative value
  - indicates sensitivity of each member of the class
  - α large and positive gives a "top-sensitive" measure
  - α negative gives a "bottom-sensitive" measure
  - each α gives a specific distance concept

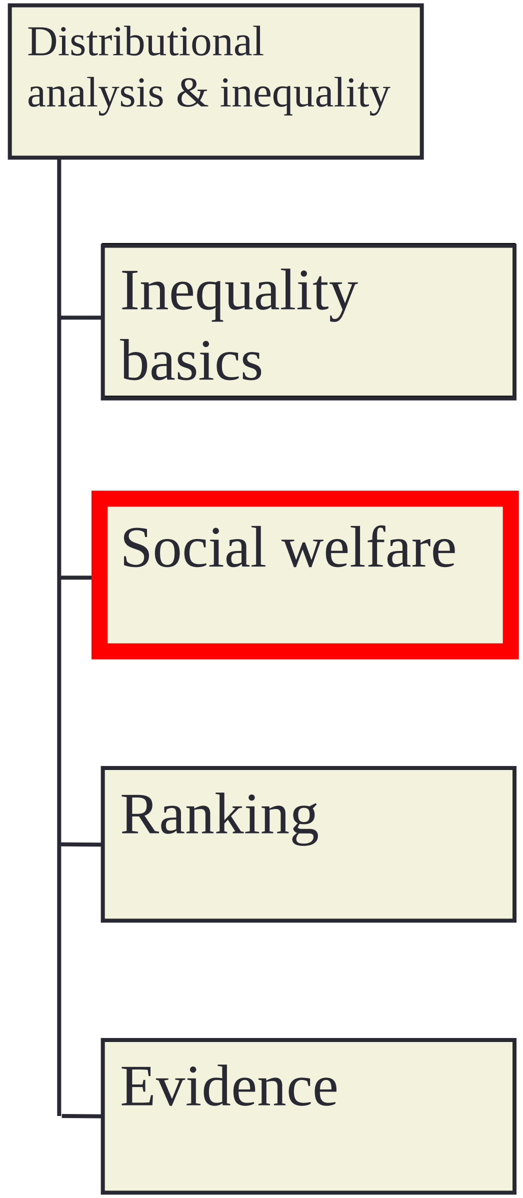
Generalised Entropy



Scale or translation independence?



Overview...

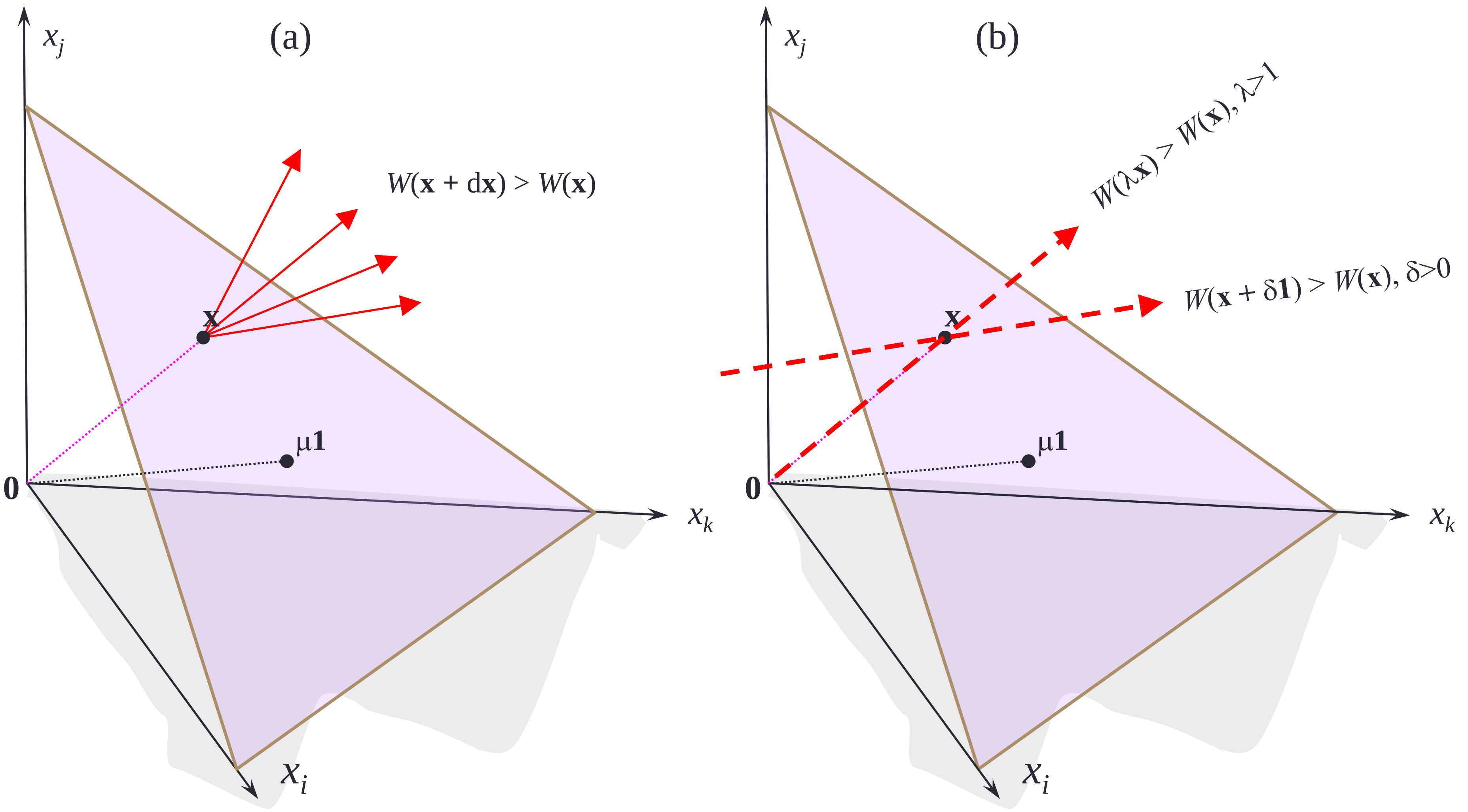


Connecting with Social values

Social-welfare functions

- A standard approach to a method of assessment
- Basic tool is a *social welfare function* (SWF)
  - Maps set of distributions into the real line $W = W(\mathbf{x})$
  - I.e. for each distribution we get one specific number
- Properties will depend on economic principles
- Simple example of a SWF: $W = \Sigma_\iota\, x_i$
- Principles on which SWF could be based?
  - use counterparts of inequality axioms
  - "reverse them" so welfare increases as inequality decreases
  - also…
- **Monotonicity.** $W(x_1,x_2..., x_i+\delta,..., x_n) > W(x_1,x_2,..., x_i,..., x_n)$

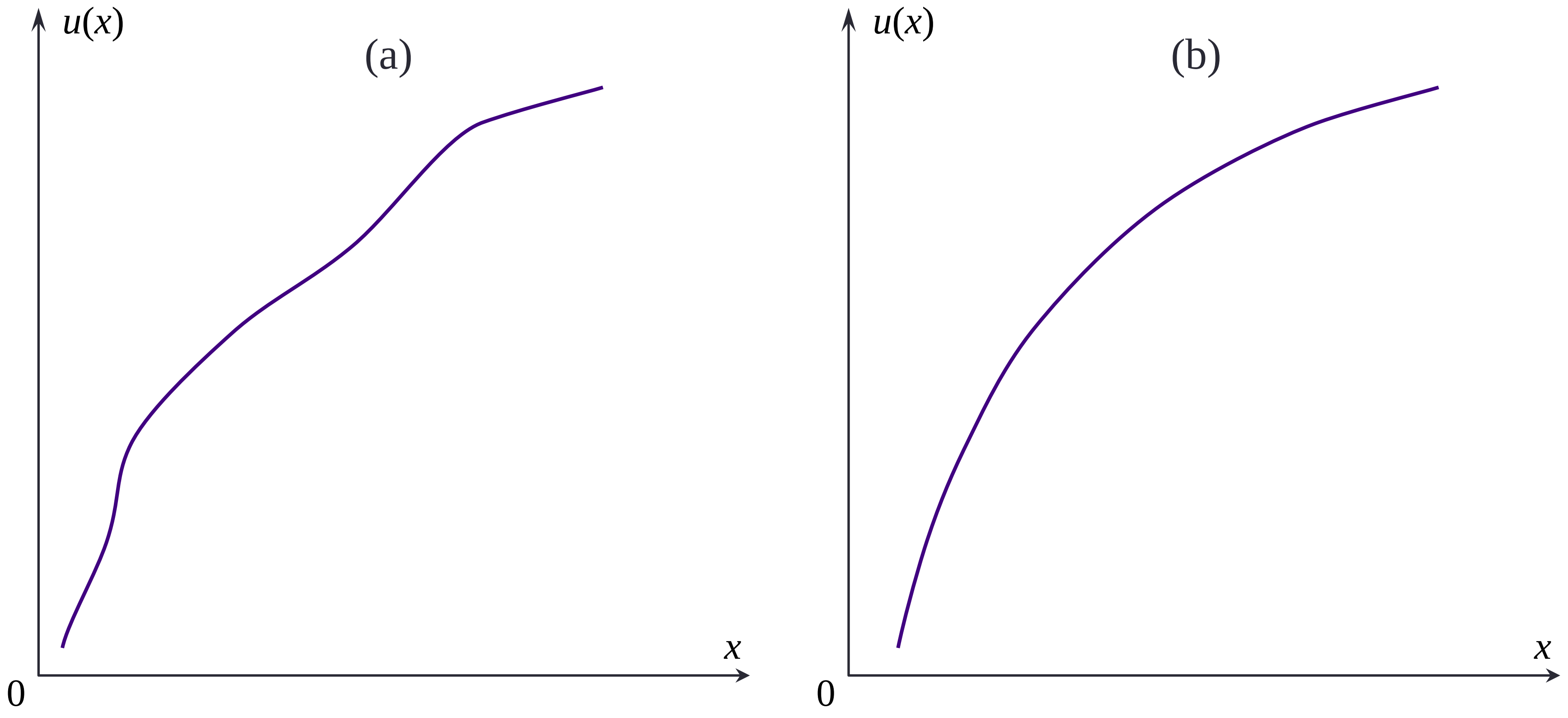
Social welfare and income growth



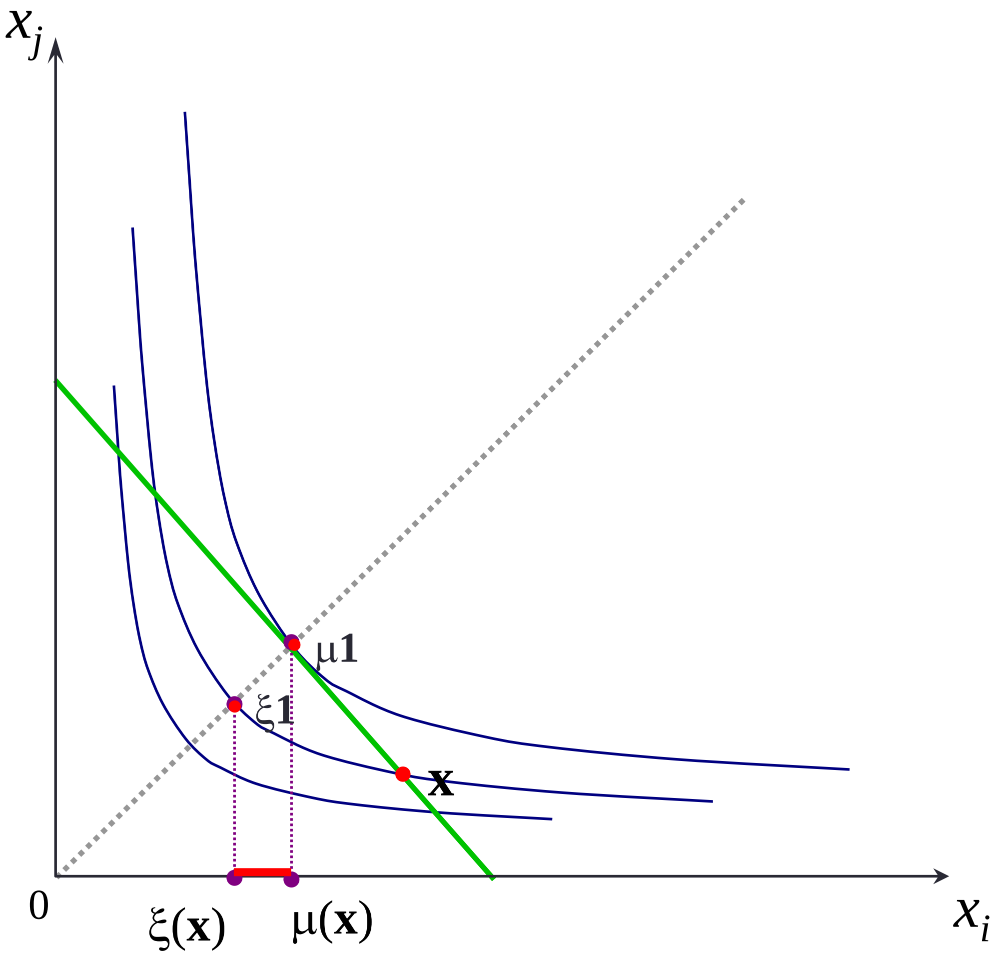
Classes of SWFs

- **Anonymity** and **population** principle:
  - can write SWF in either Irene-Janet form or *F* form
  - may need to standardise for needs etc
- Introduce **decomposability**
  - get class of Additive SWFs $\mathfrak{W}$:
  - $W(\mathbf{x}) = \Sigma_\iota\, u(x_i)$
  - or equivalently $W(F) = \int u(x)\, \mathrm{d}F(x)$
- If we impose **monotonicity** we get
  - $\mathfrak{W}_1 \subset \mathfrak{W}$: $u(\bullet)$ increasing
- If we further impose the **transfer principle** we get
  - $\mathfrak{W}_2 \subset \mathfrak{W}_1$: $u(\bullet)$ increasing and concave

Evaluation functions *u*



SWF and inequality



- *The Irene &Janet diagram*
- *A given distribution*
- *Distributions with same mean*
- *Contours of the SWF*
- *Construct an equal distribution with same social welfare*
- *Equally-Distributed Equivalent income*
- *Social waste from inequality*

- contour: **x** values such that $W(\mathbf{x})$ = const
- Curvature of contour indicates society's willingness to tolerate "efficiency loss" in pursuit of greater equality
- Inequality $1 - \frac{\xi(\mathbf{x})}{\mu(\mathbf{x})}$.

An important family

- Take the W_2 subclass and impose **scale invariance.**
- Get the family of SWFs where *u* is iso-elastic:

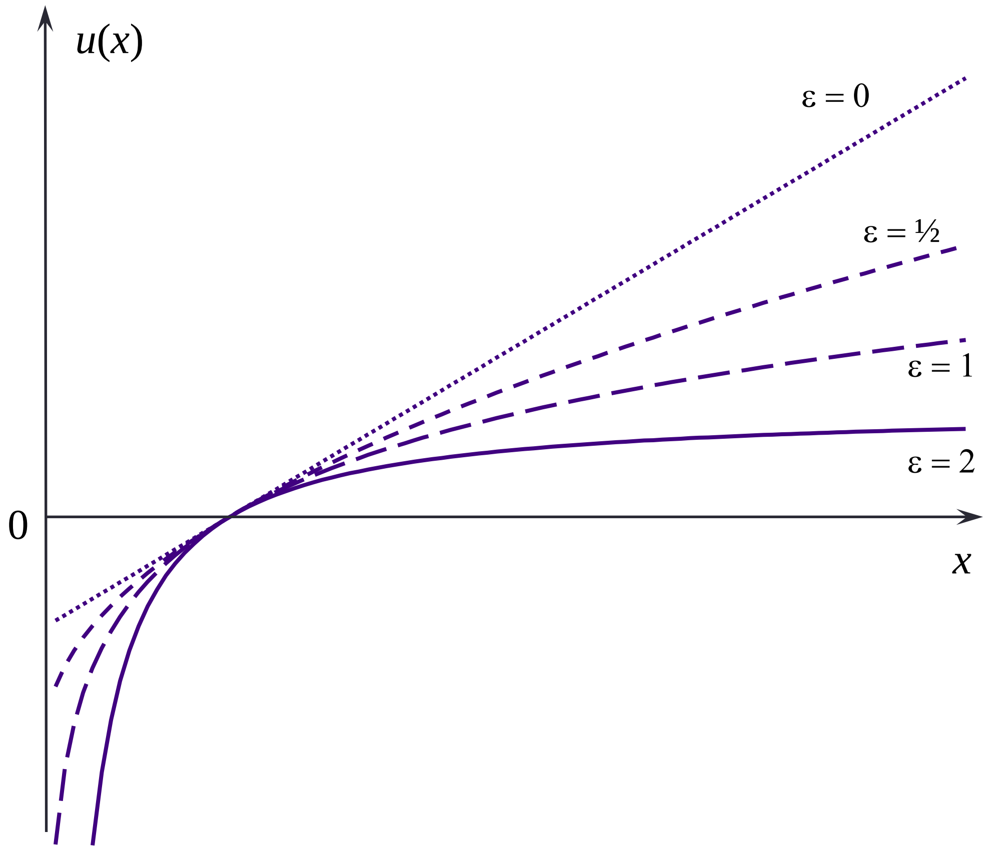
$$u(x) = \frac{x^{1-\varepsilon} - 1}{1-\varepsilon}, \quad \varepsilon \geq 0$$

  - has same form as CRRA utility function
- Parameter ε captures society's inequality aversion.
  - Similar to individual risk aversion (Atkinson 1970)

$$\xi(\mathbf{x}) = \left[\frac{1}{n}\sum_{i=1}^{n} x_i^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}, \varepsilon > 0$$

$$I_{\mathrm{A}}^{\varepsilon}(\mathbf{x}) := 1 - \left[\frac{1}{n}\sum_{i=1}^{n}\left[\frac{x_i}{\mu(\mathbf{x})}\right]^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}.$$

Isoelastic *u* for different values of ε



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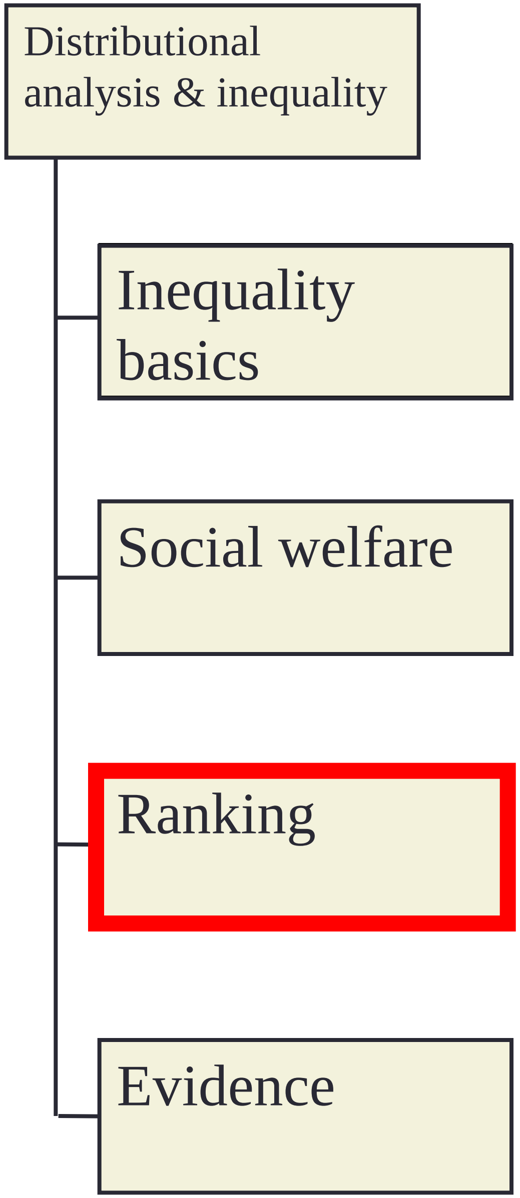


Alternative approaches within Distributional Analysis

Ranking and dominance

- Introduce two simple concepts
  - first illustrate using the Irene-Janet representation
  - take income vectors **x** and **y** for a given *n*
- First-order dominance:
  - $y_{[1]} > x_{[1]}, \quad y_{[2]} > x_{[2]}, \quad y_{[3]} > x_{[3]}$
  - Each ordered income in **y** larger than that in **x**
- Second-order dominance:
  - $y_{[1]} > x_{[1]}, \; y_{[1]}+y_{[2]} > x_{[1]}+x_{[2]}, \quad y_{[1]}+y_{[2]} +\ldots+ y_{[n]} > x_{[1]}+x_{[2]} \ldots+ x_{[n]}$
  - Each cumulated income sum in **y** larger than that in **x**
- Generalise this a little
  - represent distributions in *F*-form (anonymity, population principle)
  - q: population proportion ($0 \leq q \leq 1$)
  - $F(x)$: proportion of population with incomes $\leq x$
  - $\mu(F)$: mean of distribution F

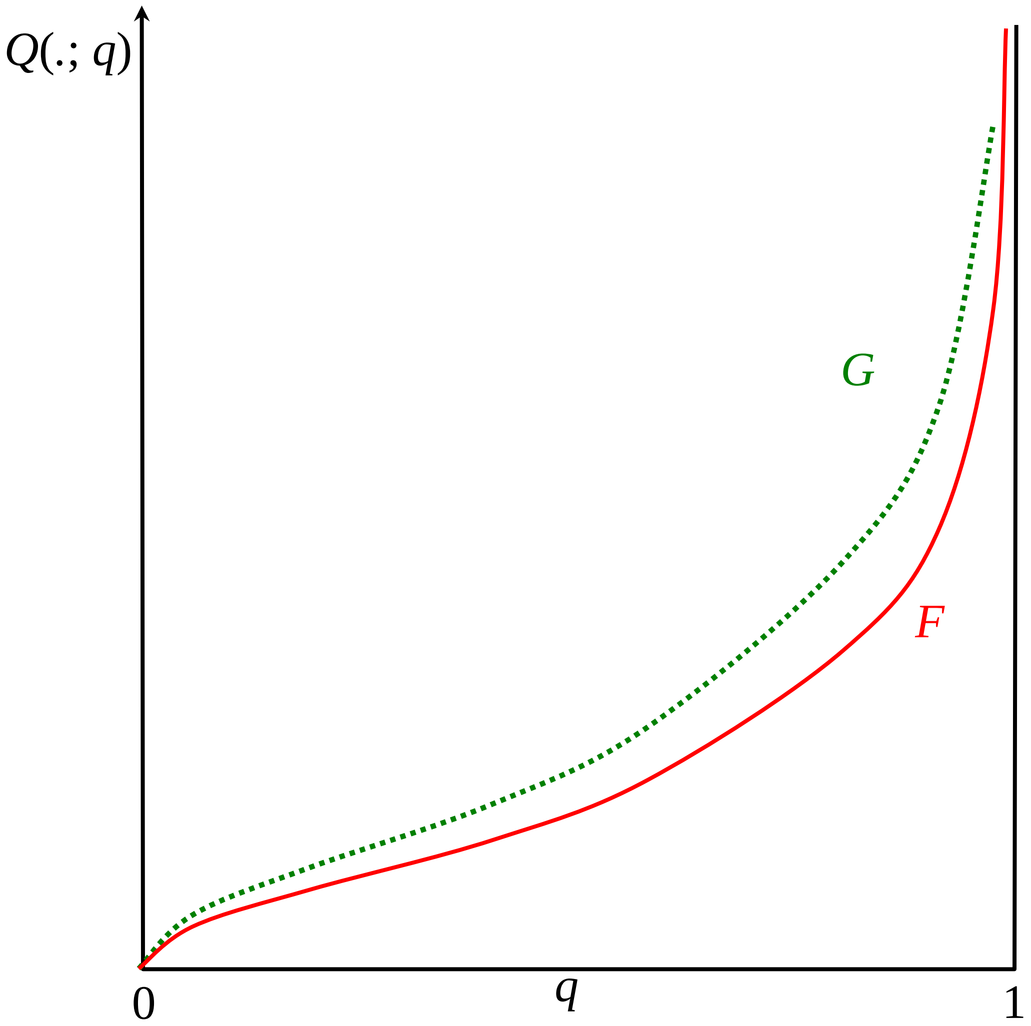
1^{st}-Order approach

- Basic tool is the *quantile*, expressed as

  $$Q(F;\, q) \;:=\; \inf\,\{x \mid F(x) \geq q\} \;=\; x_q$$

  - "smallest income such that cumulative frequency is at least as great as *q*"
- Use this to derive a number of intuitive concepts
- Also to characterise the idea of 1^{st}-order (quantile) dominance:
  - "*G* quantile-dominates *F*" means:
    - for every q, $Q(G;q) \geq Q(F;q)$,
    - for some q, $Q(G;q) > Q(F;q)$
- A fundamental result:
  - *G* quantile-dominates *F* iff $W(G) > W(F)$ for all $W \in \mathfrak{W}_1$

Parade and 1st-order dominance



- *Plot quantiles against proportion of population*
- *Parade for distribution F again*
- *Parade for distribution G*

- In this case *G* clearly quantile-dominates *F*
- But (as often happens) what if it doesn't?
- Try second-order method

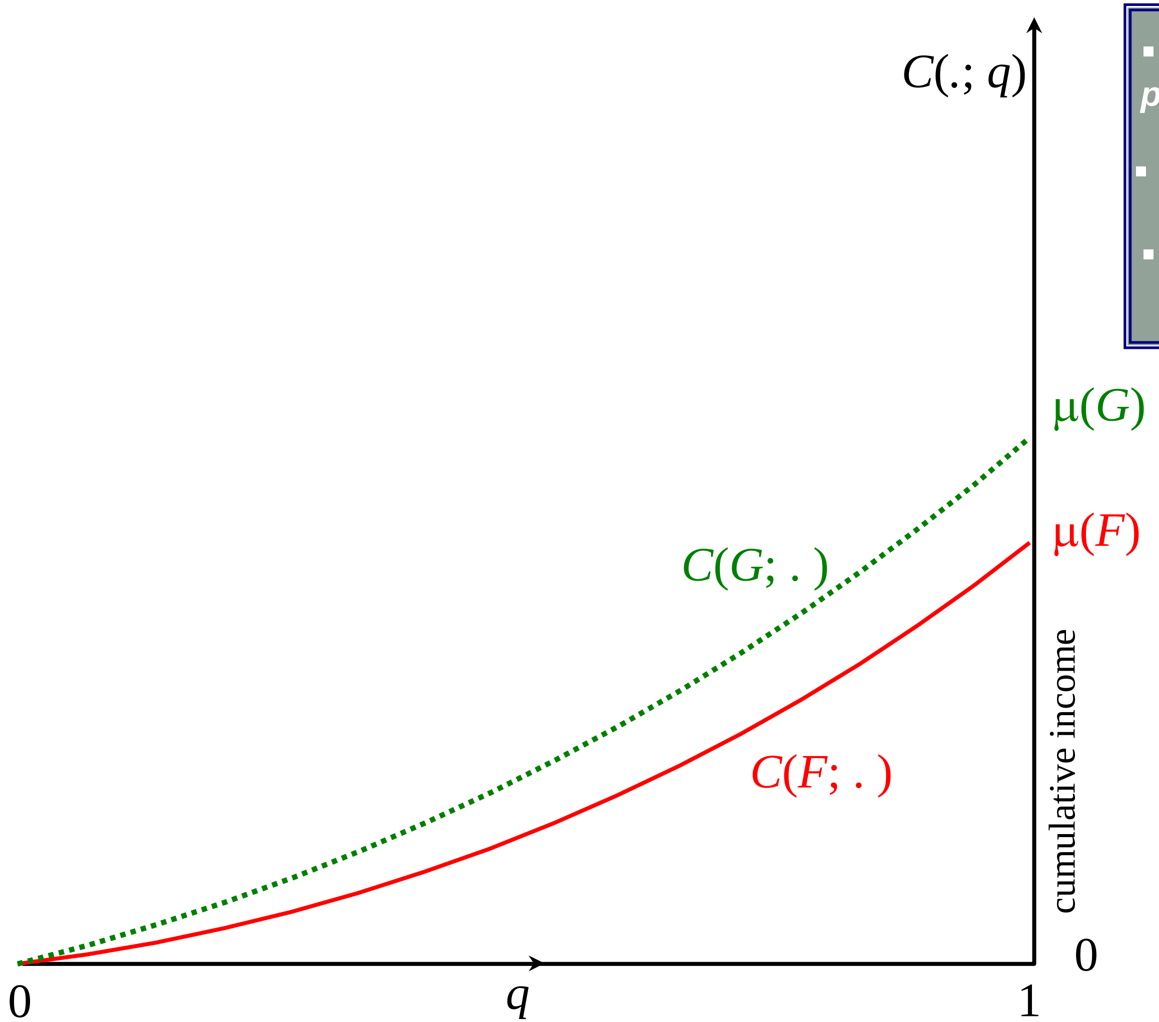
2^{nd}-Order approach

- Basic tool is the *income cumulant*, expressed as

  $$C(F;\, q) \;:= \int^{Q(F;\,q)} x \,\mathrm{d}F(x)$$

  - "The sum of incomes in the Parade, up to and including position q"
- Use this to derive a number of intuitive concepts
  - the "shares" ranking, Gini coefficient
  - graph of C the *generalised Lorenz curve*
- Also to characterise the idea of 2^{nd}-order (cumulant) dominance:
  - "G cumulant-dominates F" means:
    - for every q, $C(G;q) \geq C(F;q)$,
    - for some q, $C(G;q) > C(F;q)$
- A fundamental result (Shorrocks 1983):
  - G cumulant-dominates F iff $W(G) > W(F)$ for all $W \in \mathfrak{W}_2$

GLC and 2nd-order dominance



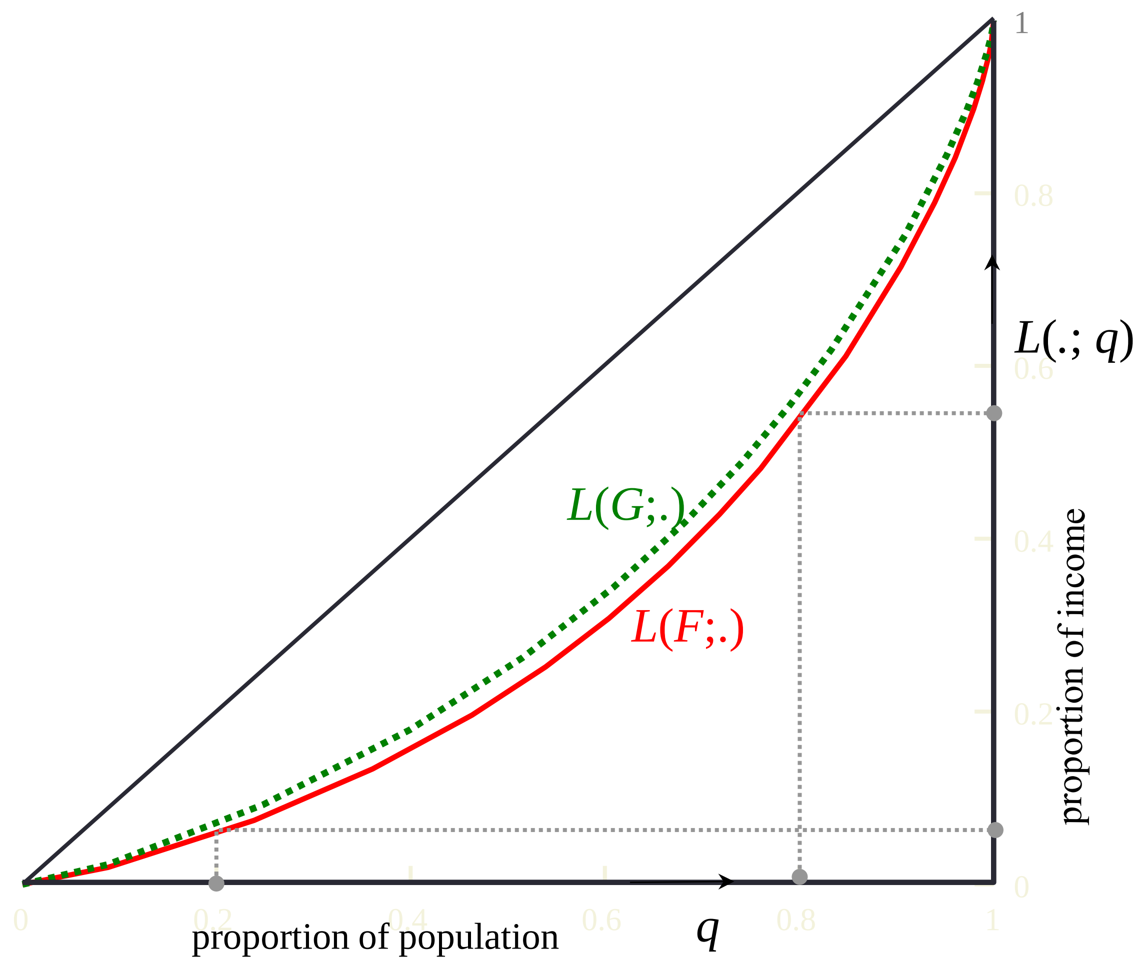
- Plot cumulations against proportion of population
- GLC for distribution F
- GLC for distribution G

- Intercept on vertical axis is at mean income

2nd-Order approach (continued)

- The *share* of the proportion q of distribution F is $L(F;q) := C(F;q) / \mu(F)$
  - "income cumulation at q divided by total income"
- Yields Lorenz dominance, or the "shares" ranking:
  - "G Lorenz-dominates F" means:
    - for every q, $L(G;q) \geq L(F;q)$,
    - for some q, $L(G;q) > L(F;q)$
- Another fundamental result (Atkinson 1970):
  - For given μ, G Lorenz-dominates F iff $W(G) > W(F)$ for all $W \in \mathfrak{W}_2$

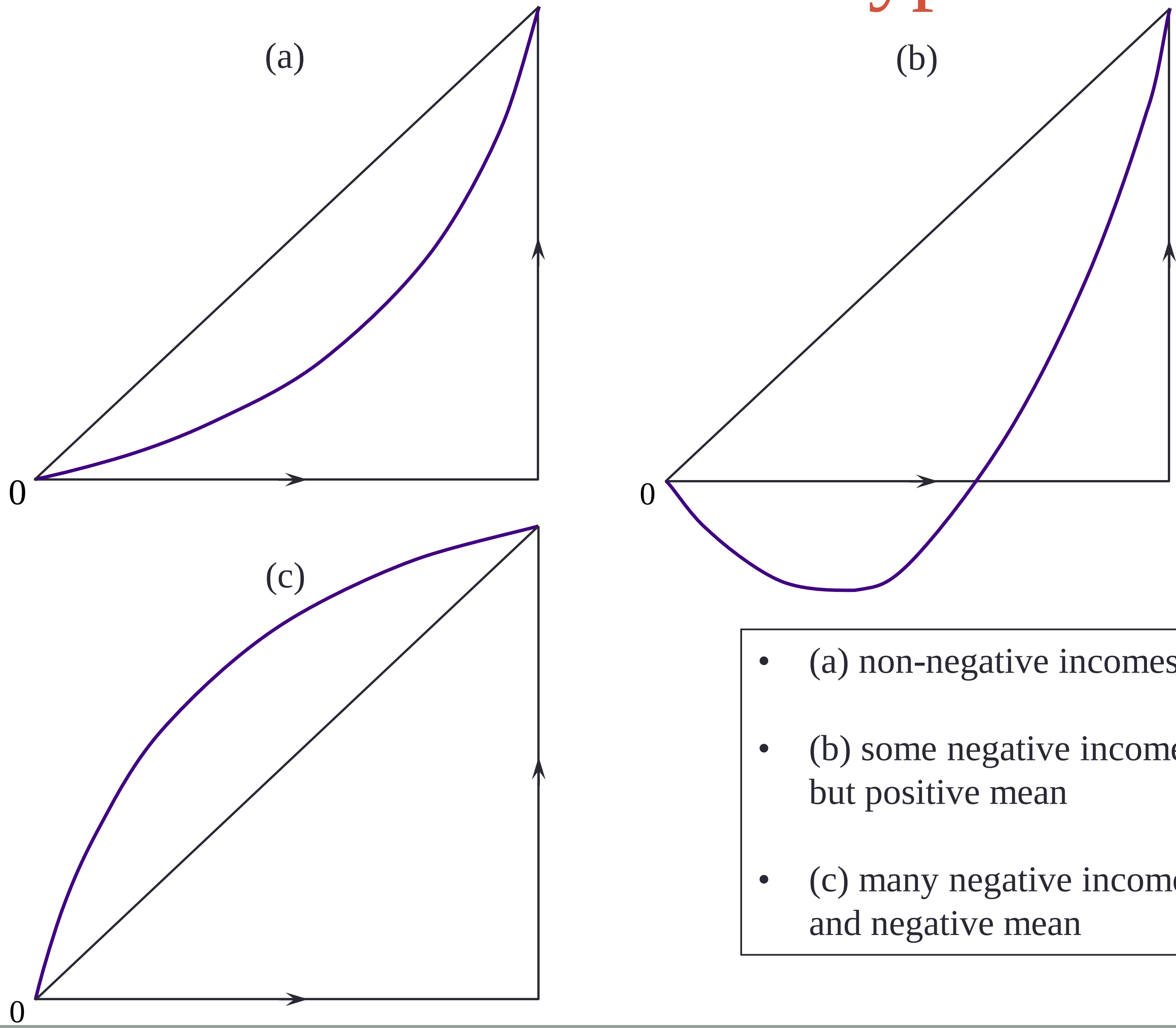
Lorenz curve and ranking



- *Plot shares against proportion of population*
- *Perfect equality*
- *Lorenz curve for distribution F*
- *Lorenz curve for distribution G*

- In this case *G* clearly Lorenz-dominates *F*
- So *F* displays more inequality than *G*
- But what if L-curves intersect?
- No clear statement about inequality (or welfare) is possible without further information

Lorenz curves for different types of data



- (a) non-negative incomes
- (b) some negative incomes, but positive mean
- (c) many negative incomes and negative mean

Overview...

Distributional analysis & inequality

- Inequality basics
- Social welfare
- Ranking
- **Evidence**

Attitudes and perceptions

Views on distributions

- Do people make distributional comparisons in the same way as economists?
- Summarised from Amiel-Cowell (1999)
  - examine proportion of responses in conformity with standard axioms
  - both directly in terms of inequality and in terms of social welfare

	Inequality		*SWF*	
	Num	*Verbal*	*Num*	*Verbal*
Anonymity	83%	72%	66%	54%
Population	58%	66%	66%	53%
Decomposability	57%	40%	58%	37%
Monotonicity	-	-	54%	55%
Transfers	35%	31%	47%	33%
Scale indep.	51%	47%	-	-

Inequality aversion

- Are people averse to inequality?
  - evidence of both inequality and risk aversion (Carlsson et al 2005)
  - risk-aversion as proxy for inequality aversion? (Cowell and Gardiner 2000)
- What value for ε?
  - affected by way the question is put? (Pirttilä and Uusitalo 2010)
  - high values of risk aversion from survey evidence Barsky et al 1997)
  - lower values of risk aversion from savings analysis (Blundell et al 1994)
  - from happiness studies 1.0 to 1.5 (Layard et al 2008)
  - related to the extent of inequality in the country? (Lambert et al 2003)
  - perhaps a value of around 0.7 – 2 is reasonable (HM Treasury 2011 pp 93-94)

Conclusion

- Axiomatisation of inequality or welfare can be accomplished using just a few basic principles
- Ranking criteria can provide broad judgments
- But may be indecisive, so specific SWFs could be used
  - What shape should they have?
  - How do we specify them empirically?
- Several axioms survive scrutiny in experiment
  - but Transfer Principle often rejected

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