Chapter 7

General Equilibrium

Exercise 7.1 Suppose there are 200 traders in a market all of whom behave as price takers. Suppose there are three goods and the traders own initially the following quantities:

- 100 of the traders own 10 units of good 1 each
- 50 of the traders own 5 units of good 2 each
- 50 of the traders own 20 units of good 3 each

All the traders have the utility function

\[ U = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} x_3^{\frac{1}{2}} \]

What are the equilibrium relative prices of the three goods? Which group of traders has members who are best off?

Outline Answer:

For each group of traders the Lagrangean may be written

\[ \frac{1}{2} \log x_1^h + \frac{1}{4} \log x_2^h + \frac{1}{4} \log x_3^h + v^h [y^h - p_1 x_1^h - p_2 x_2^h - p_3 x_3^h] \]

where \( h = 1, 2, 3 \) and \( y^1 = 10p_1, y^2 = 5p_2 \) and \( y^3 = 20p_3 \). From the first-order conditions we find that for a trader of type \( h \):

\[ x_1^h = \frac{y^h}{2p_1} \]
\[ x_2^h = \frac{y^h}{4p_2} \]
\[ x_3^h = \frac{y^h}{4p_3} \]

Excess demand for good 1 and 2 are then:
\[ E_1 = 100x_1^1 + 50x_1^2 + 50x_1^3 - 100 \]
\[ E_2 = 100x_2^1 + 50x_2^2 + 50x_2^3 - 250 \]

Substituting in for \( x_i^h \) and \( y^h \) and putting \( E_1 = E_2 = 0 \) we find

\[-500 + \frac{125p_2}{p_1} + \frac{500p_3}{p_1} = 0\]
\[\frac{250p_1}{p_2} - \frac{750}{4} + \frac{250p_3}{p_2} = 0\]

that implies \( \frac{p_2}{p_1} = 2 \) and \( \frac{p_3}{p_1} = \frac{1}{2} \).

Using good 1 as numeraire we immediately see that \( y^1 = y^2 = y^3 = 10 \). All are equally well off.
Exercise 7.2 Consider an exchange economy with two goods and three persons. Alf always demands equal quantities of the two goods. Bill’s expenditure on group 1 is always twice his expenditure on good 2. Charlie never uses good 2.

1. Describe the indifference maps of the three individuals and suggest utility functions consistent with their behaviour.

2. If the original endowments are respectively (5, 0), (3, 6) and (0, 4), compute the equilibrium price ratio. What would be the effect on equilibrium prices and utility levels if

(a) 4 extra units of good 1 were given to Alf;
(b) 4 units of good 1 were given to Charlie?

Outline Answer:

1. Let $\rho = \frac{p_1}{p_2}$ so that values are measured in terms of good 2.

(a) Alf’s (binding) budget constraint is

$$\rho x_1^a + x_2^a = 5\rho$$

Therefore, given the information in the question, the demand functions are

$$x_1^a = x_2^a = \frac{5\rho}{\rho + 1}.$$

The utility function consistent with this behaviour is

$$U^a(x_1^a, x_2^a) = \min \{x_1^a, x_2^a\}$$

– see Figure 7.1

(b) Bill’s budget constraint is

$$\rho x_1^b + x_2^b = 3\rho + 6$$

From the question we have

$$\rho x_1^b = 2x_2^b$$

Therefore:

$$x_1^b = 2 + \frac{4}{\rho}$$

$$x_2^b = \rho + 2.$$ 

The utility function is Cobb-Douglas:

$$U^b = 2 \log x_1^b + \log x_2^b$$

– see Figure 7.2

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Figure 7.1: Alf’s preferences and demand

Figure 7.2: Bill’s preferences and demand
(c) Charlie’s budget constraint is
\[ \rho x_1^a + x_2^a = 4 \]
Given the information in the question we have
\[ x_1^c = 4/\rho \]
\[ x_2^c = 0 \]
and utility is
\[ U^c = x_1^c \]
– see Figure 7.3

Figure 7.3: Charlie’s preferences and demand

2. Excess demand for good 2 is:
\[ E_2 = \frac{5\rho}{\rho + 1} + \rho + 2 - 10. \]
Putting \( E_2 = 0 \) yields \( \rho = -2 \) or 4. Hence the equilibrium price ratio is 4. Utility levels are \( U^a = 4, U^b = \log(54) \) and \( U^c = 1 \).

(a) Excess demand is now
\[ \frac{9\rho}{\rho + 1} + \rho + 2 - 10 \]
and the equilibrium price ratio is 2. Utility levels are \( U^a = 6, U^b = \log(64) \) and \( U^c = 2 \).

(b) Excess demand for good 2 is:
\[ E_2 = \frac{5\rho}{\rho + 1} + \rho + 2 - 10. \]
Putting \( E_2 = 0 \) yields \( \rho = -2 \) or 4. Hence the equilibrium price ratio is 4. Utility levels are \( U^a = 4, U^b = \log(54) \) and \( U^c = 4 + 1 = 5 \).
Exercise 7.3 In a two-commodity economy assume a person has the endowment \((0, 20)\).

1. Find the person’s demand function for the two goods if his preferences are represented by each of the types A to D in Exercise 4.2. In each case explain what the offer curve must look like.

2. Assume that there are in fact two equal sized groups of people, each with preferences of type A, where everyone in group 1 has the endowment \((10, 0)\) with \(\alpha = \frac{1}{2}\) and everyone in group 2 an endowment \((0, 20)\) with \(\alpha = \frac{3}{4}\). Use the offer curves to find the competitive equilibrium price and allocation.

Outline Answer:

1. The income of person \(h\) is 20.

   (a) If he has preferences of type A then the Lagrangean is
   \[
   \alpha \log x_1^h + [1 - \alpha] \log x_2^h + \lambda \left[20 - \rho x_1^h - x_2^h\right] \tag{7.1}
   \]
   First order conditions for an interior maximum of \((7.1)\) are
   \[
   \frac{\alpha}{x_1^h} - \lambda \rho = 0 \\
   \frac{1 - \alpha}{x_2^h} - \lambda = 0 \\
   20 - \rho x_1^h - x_2^h = 0
   \]
   Solving these we find \(\lambda = \frac{1}{20}\) and so the demands will be
   \[
   x^h = \begin{bmatrix}
   \frac{20\alpha}{\rho} \\
   20[1 - \alpha]
   \end{bmatrix}
   \tag{7.2}
   \]
   and the offer curve will simply be a horizontal straight line at \(x_2^h = 20[1 - \alpha]\).

   (b) If \(h\) has preferences of type B then demand will be
   \[
   x^h = \begin{cases} 
   x', & \text{if } \rho > \beta \\
   x^h \in [x', x''], & \text{if } \rho = \beta \\
   (20/\rho, 0), & \text{if } \rho < \beta
   \end{cases}
   \]
   where \(x' := (0, 20), x'' := (20/\beta, 0)\), and their offer curve will consist of the union of the line segment \([x', x'']\) and the line segment from \(x''\) to \((\infty, 0)\).

   (c) If group-2 persons have preferences of type C then their demands will be
   \[
   x^h = \begin{cases} 
   x', & \text{if } \rho > \sqrt{7} \\
   x' \text{ or } x'', & \text{if } \rho = \sqrt{7} \\
   (20/\rho, 0), & \text{if } \rho < \sqrt{7}
   \end{cases}
   \]
   where \(x' := (0, 20), x'' := (20/\sqrt{7}, 0)\), and their offer curve will consist of the union of the point \(x'\) and the line segment from \(x''\) to \((\infty, 0)\).
(d) If group-2 persons have preferences of type D then their demands will be

\[ x^h = \begin{bmatrix} 20 \\ \rho + 3 \\ 20 \rho + 3 \end{bmatrix} \]

and their offer curve is just the straight line \( x^h_2 = \delta x^h_1 \). These are illustrated in Figure 7.4.

2. If a type-A person had an income of 10 units of commodity 1 then, by analogy with part 1, demand would be

\[ x^1 = \begin{bmatrix} 10 \alpha \\ 10 \rho [1 - \alpha] \end{bmatrix} \]

and the offer curve will simply be a vertical straight line at \( x^h_1 = 10 \alpha \). From (7.2) and (7.3) we have \( x^1_1 = 10 \cdot \frac{1}{2} = 5 \), \( x^1_2 = 20 \cdot \frac{3}{4} = 5 \). Given that there are 10 units per person of commodity 1 and 20 units per person of commodity 2 the materials balance condition then means that
the equilibrium allocation must be

\[
x^1 = \begin{bmatrix} 5 \\ 15 \end{bmatrix}
\]

\[
x^2 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}
\]

Solving for \( \rho \) from (7.2) and (7.3) we find that the equilibrium price ratio must be 3.
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Exercise 7.4 The agents in a two-commodity exchange economy have utility functions

\[ U^a(x^a) = \log(x_1^a) + 2 \log(x_2^a) \]
\[ U^b(x^b) = 2 \log(x_1^b) + \log(x_2^b) \]

where \( x^h_i \) is the consumption by agent \( h \) of good \( i \), \( h = a, b; i = 1, 2 \). The property distribution is given by the endowments \( R^a = (9, 3) \) and \( R^b = (12, 6) \).

1. Obtain the excess demand function for each good and verify that Walras’ Law is true.
2. Find the equilibrium price ratio.
3. What is the equilibrium allocation?
4. Given that total resources available remain fixed at \( R := R^a + R^b = (21, 9) \), derive the contract curve.

Outline Answer:

1. To get the demand functions for each person we need to find the utility-maximising solution. The Lagrangean for person \( a \) is

\[ L^a(x^a, \nu^a) := \log(x_1^a) + 2 \log(x_2^a) + \nu^a [9p_1 + 3p_2 - p_1x_1^a - p_2x_2^a] \]

First-order conditions are

\[ \begin{cases} \frac{1}{x_1^a} - \nu^a p_1 = 0 \\ \frac{2}{x_2^a} - \nu^a p_2 = 0 \\ 9p_1 + 3p_2 - p_1x_1^a - p_2x_2^a = 0 \end{cases} \] \hspace{1cm} \text{(7.4)}

Define \( \rho := p_1/p_2 \) and normalise \( p_2 \) arbitrarily at 1. Then, rearranging the FOC we get

\[ \begin{cases} \frac{1}{x_1^a} = \rho x_2^a \\ \frac{2}{x_2^a} = x_2^a \\ 9\rho + 3 = \rho x_1^a + x_2^a \end{cases} \] \hspace{1cm} \text{(7.4)}

Subtracting the first two equations from the third in \textbf{(7.4)} we can see that \( \nu^a = \frac{1}{9\rho + 3} \). Substituting back for the Lagrange multiplier \( \nu^a \) into the first two parts of \textbf{(7.4)} we see that the first-order conditions imply:

\[ \begin{bmatrix} x_1^a \\ x_2^a \end{bmatrix} = \begin{bmatrix} 3 + \frac{1}{\rho} \\ 6\rho + 2 \end{bmatrix} \] \hspace{1cm} \text{(7.5)}

Using exactly the same method for person \( b \) we would find

\[ \begin{bmatrix} x_1^b \\ x_2^b \end{bmatrix} = \begin{bmatrix} 8 + \frac{4}{\rho} \\ 4\rho + 2 \end{bmatrix} \] \hspace{1cm} \text{(7.6)}

Using the definition we can then find the excess demand functions by evaluating:

\[ E_i := x_1^{*a} + x_1^{*b} - R_i^a - R_i^b \] \hspace{1cm} \text{(7.7)}

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Doing this we get
\[
\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{p} - 10 \\ 10\rho - 5 \end{bmatrix}
\] (7.8)

Now construct the weighted sum of excess demands. It is obvious that
\[\rho E_1 + E_2 = 0\] (7.9)

thus confirming Walras’ Law. In equilibrium the materials’ balance condition must hold and so excess demand for each good must be zero, unless the corresponding equilibrium price is zero (markets clear).

2. Solving for \(E_1 = 0\) in (7.8) we find \(\rho^* = \frac{1}{2}\) for the (normalised) equilibrium prices.

3. The allocation is
\[
\begin{bmatrix} x_1^b \\ x_2^b \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \end{bmatrix}
\]

4. The contract curve is traced out by the MRS condition
\[\text{MRS}^a_{12} = \text{MRS}^b_{12}\] (7.10)

and the materials balance condition
\[\mathbf{E} = 0\] (7.11)

From (7.11) we have
\[
\begin{bmatrix} x_1^b \\ x_2^b \end{bmatrix} = \begin{bmatrix} 21 - x_1^a \\ 9 - x_2^a \end{bmatrix}
\] (7.12)

Applying (7.10) we then get
\[
\frac{2x_1^a}{x_2^a} = \frac{21 - x_1^a}{2[9 - x_2^a]}
\] (7.13)

which implies that the equation of the contract curve is:
\[x_2^a = \frac{12x_1^a}{x_1^a + 7}.
\] (7.14)
Exercise 7.5 Which of the following sets of functions are legitimate excess demand functions?

\[
\begin{align*}
E_1(p) &= -p_2 + \frac{10}{p_1} \\
E_2(p) &= p_1 \\
E_3(p) &= -\frac{10}{p_3}
\end{align*}
\]  
\tag{7.15}

\[
\begin{align*}
E_1(p) &= \frac{p_2 + p_3}{p_2} \\
E_2(p) &= \frac{p_1 + p_3}{p_2} \\
E_3(p) &= \frac{p_1 + p_2}{p_3}
\end{align*}
\]  
\tag{7.16}

\[
\begin{align*}
E_1(p) &= \frac{p_3}{p_1} \\
E_2(p) &= \frac{p_3}{p_2} \\
E_3(p) &= -2
\end{align*}
\]  
\tag{7.17}

Outline Answer:

The first system is not homogeneous of degree zero in prices. The second violates Walras’ Law. The third one is both homogeneous and satisfies Walras’ Law.
Exercise 7.6 In a two-commodity economy let $\rho$ be the price of commodity 1 in terms of commodity 2. Suppose the excess demand function for commodity 1 is given by

$$1 - 4\rho + 5\rho^2 - 2\rho^3.$$ 

How many equilibria are there? Are they stable or unstable? How might your answer be affected if there were an increase in the stock of commodity 1 in the economy?

Outline Answer:
The excess demand for commodity 1 at relative price $\rho$ can be written

$$E(\rho) := 1 - 4\rho + 5\rho^2 - 2\rho^3 = [1 - \rho]^2(1 - 2\rho).$$

So that

$$dE(\rho)/d\rho = -4 + 10\rho - 6\rho^2$$

–see Figure 7.5. From this we see that there are two equilibria as follows:

1. $\rho = 0.5$. Here $dE(\rho)/d\rho < 0$ and so it is clear that the equilibrium is locally stable.

2. $\rho = 1$. Here $dE(\rho)/d\rho = 0$. But the graph of the function reveals that it is locally stable from “above” (where $\rho > 1$) and unstable from “below” (where $\rho < 1$).

If there were an increase in the stock of commodity 1 the excess demand function would be shifted to the left in Figure 7.5 – then there is only one, stable equilibrium.

![Figure 7.5: Excess demand](image-url)
Exercise 7.7 Consider the following four types of preferences:

Type A : $\alpha \log x_1 + [1 - \alpha] \log x_2$
Type B : $\beta x_1 + x_2$
Type C : $\gamma [x_1]^2 + [x_2]^2$
Type D : $\min \{\delta x_1, x_2\}$

where $x_1, x_2$ denote respectively consumption of goods 1 and 2 and $\alpha, \beta, \gamma, \delta$ are strictly positive parameters with $\alpha < 1$.

1. Draw the indifference curves for each type.

2. Assume that a person has an endowment of 10 units of commodity 1 and zero of commodity 2. Show that, if his preferences are of type A, then his demand for the two commodities can be represented as

$$x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10\alpha \\ 10\rho[1 - \alpha] \end{bmatrix}$$

where $\rho$ is the price of good 1 in terms of good 2. What is the person’s offer curve in this case?

3. Assume now that a person has an endowment of 20 units of commodity 2 (and zero units of commodity 1) and find the person’s demand for the two goods if his preferences are represented by each of the types A to D. In each case explain what the offer curve must look like.

4. In a two-commodity economy there are two equal-sized groups of people. People in group 1 own all of commodity 1 (10 units per person) and people in group 2 own all of commodity 2 (20 units per person). If Group 1 has preferences of type A with $\alpha = \frac{1}{2}$ find the competitive equilibrium prices and allocations in each of the following cases:

(a) Group 2 have preferences of type A with $\alpha = \frac{3}{4}$
(b) Group 2 have preferences of type B with $\beta = 3$.
(c) Group 2 have preferences of type D with $\delta = 1$.

5. What problem might arise if group 2 had preferences of type C? Compare this case with case 4b.

Outline Answer:

1. Indifference curves have the shape shown in the figure 7.6

2. The income of a group-1 person is $10\rho$. If group-1 persons have preferences of type A then the Lagrangean is

$$\alpha \log x_1^1 + [1 - \alpha] \log x_1^2 + \lambda \left[10\rho x_1^1 - x_2^1\right]$$
First order conditions for an interior maximum are

\[ \frac{\alpha}{x_1^1} - \lambda \rho = 0 \]
\[ \frac{1 - \alpha}{x_2^1} - \lambda = 0 \]
\[ 10\rho - \rho x_1^1 - x_2^1 = 0 \]

Solving these we find \( \lambda = \frac{1}{10\rho} \) and so the demands will be

\[ x^1 = \left[ \begin{array}{c} 10\alpha \\ 10\rho [1 - \alpha] \end{array} \right] \]

and the offer curve will simply be a vertical straight line at \( x_1^1 = 10\alpha \).

3. The income of a group-2 person is 20. So, if group-2 persons have preferences of type A, then their demands will be

\[ x^2 = \left[ \begin{array}{c} 20\alpha \\ 20 [1 - \alpha] \end{array} \right] \]

and their offer curve will simply be a horizontal straight line at \( x_2^2 = 20 [1 - \alpha] \). If group-2 persons have preferences of type B then their demands will be

\[ x^2 = \begin{cases} x', & \text{if } \rho > \beta \\ [x', x''], & \text{if } \rho = \beta \\ (20/\rho, 0), & \text{if } \rho < \beta \end{cases} \]
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where $x' := (0, 20)$, $x'' := (20/\beta, 0)$, and their offer curve will consist of the union of the line segment $[x', x'']$ and the line segment from $x''$ to $(\infty, 0)$. If group-2 persons have preferences of type C then their demands will be

$$x^2 = x', \text{ if } \rho > \sqrt{\gamma}$$

$$x^2 = x' \text{ or } x'', \text{ if } \rho = \sqrt{\gamma}$$

$$x^2 = (20/\rho, 0), \text{ if } \rho < \sqrt{\gamma}$$

where $x' := (0, 20)$, $x'' := (20/\sqrt{\gamma}, 0)$, and their offer curve will consist of the union of the point $x'$ and the line segment from $x''$ to $(\infty, 0)$. If group-2 persons have preferences of type D then their demands will be

$$x^2 = \begin{bmatrix} \frac{20}{\rho+\delta} \\ \frac{20\delta}{\rho+\delta} \end{bmatrix}$$

and their offer curve is just the straight line $x_2^2 = \delta x_1^2$.

4. In each case below we could work out the excess demand function, set excess demand equal to zero, find the equilibrium price and then the equilibrium allocation. However, we can get to the result more quickly by using an equivalent approach. Given that an equilibrium allocation must lie at the intersection of the offer curves of the two parties the answer in each case is immediate.

(a) From the above computations we have $x_1^1 = 10 \cdot \frac{1}{2} = 5$, $x_2^2 = 20 \left[ 1 - \frac{1}{3} \right] = 5$. Given that there are 10 units per person of commodity 1 and 20 units per person of commodity 2 the materials balance condition then means that the equilibrium allocation must be

$$x^1 = \begin{bmatrix} 5 \\ 15 \end{bmatrix}$$

$$x^2 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

Solving for $\rho$ we find that the equilibrium price ratio must be 3.

(b) We have $x_1^1 = 5$, and so $x_2^1 = 5$. Using the fact that the equilibrium must lie on the group-2 offer curve we see that the solution must lie on the straight line from $(0, 20)$ to $(20/3, 0)$ we find that $x_2^2 = 5$ and, from the materials balance condition $x_1^2 = 20 - 5 = 15$ (as in the previous case). By the same reasoning as in the previous case the equilibrium price must be $\rho = 3$.

(c) Once again we have $x_1^1 = 5$, and so $x_2^1 = 5$. Given that the group-2 offer curve in this case is such that the person always consumes equal quantities of the two goods we must have $x_2^2 = 5$ and so again $x_2^2 = 20 - 5 = 15$ (as in the previous cases). As before the equilibrium price must be 3.

5. Note that the demand function and the offer curve for the group-2 people is discontinuous. So, if there are relatively small numbers in each group

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there may be no equilibrium (the two offer curves do not intersect). In the large numbers case we could appeal to a continuity argument and have an equilibrium with proportion of group 2 at point $x'$ and the rest at $x''$. The equilibrium would then look very much like case [15].
Exercise 7.8 In a two-commodity exchange economy there are two equal-sized groups of people. Those of type a have the utility function
\[ U_a(x^a) = \frac{1}{2} [x^a_1]^{-2} - \frac{1}{2} [x^a_2]^{-2} \]
and a resource endowment of \((R_1, 0)\); those of type b have the utility function
\[ U_b(x^b) = x^b_1 x^b_2 \]
and a resource endowment of \((0, R_2)\).

1. How many equilibria does this system have?

2. Find the equilibrium price ratio if \(R_1 = 5\), \(R_2 = 16\).

Outline Answer:
For consumers of type a the relevant Lagrangean is
\[ -\frac{1}{2} [x^a_1]^{-2} - \frac{1}{2} [x^a_2]^{-2} + \lambda [pR_1 - px^a_1 - x^a_2] \]
where \(p\) is the price of good 1 in terms of good 2. The FOC for a maximum are
\[ \frac{x^a_1}{x^a_2} - p \lambda = 0 \]
\[ \frac{x^a_2}{x^a_2} - \lambda = 0 \]
Rearranging and using the budget constraint we get
\[ x^a_1 = p^{-1/3} \lambda^{-1/3} \]
\[ x^a_2 = \lambda^{-1/3} \]
\[ px^a_1 + x^a_2 = \left[ p^{2/3} + 1 \right] \lambda^{-1/3} = pR_1 \]
So
\[ x^a_2 = \lambda^{-1/3} = \frac{pR_1}{p^{2/3} + 1} \]
For consumers of type b the Lagrangean is
\[ \log x^b_1 + \log x^b_2 + \mu [R_2 - px^b_1 - x^b_2] \]
The FOC for a maximum are
\[ \frac{x^b_1}{x^b_2} - \mu = 0 \]
\[ \frac{x^b_2}{x^b_2} - \mu = 0 \]
Rearranging and using the budget constraint we get
\[ x^b_1 = \frac{1}{\mu p} \]
\[ x^b_2 = \frac{1}{\mu} \]
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\[ px_1^b + x_2^b = \frac{2}{\mu} = R_2 \]

So

\[ x_2^b = \frac{1}{\mu} = \frac{R_2}{2} \]

The excess-demand function for good 2 is therefore

\[ \frac{pR_1}{p^{2/3} + 1} - \frac{R_2}{2} \]

1. Excess demand is 0 where

\[ p\theta - p^{2/3} - 1 = 0 \]

where \( \theta := \frac{2R_1}{R_2} \). This is equivalent to requiring

\[ p^{2/3} = p\theta - 1 \]

The expression \( p^{2/3} \) is an increasing concave function through the origin. It is clear that the straight line given by \( p\theta - 1 \) can cut this just once. There is one equilibrium.

2. If \( (R_1, R_2) = (10, 32) \) then \( \theta := \frac{20}{32} = \frac{5}{8} \) and \( p = 8 \).
Exercise 7.9 In a two-person, private-ownership economy persons \(a\) and \(b\) each have utility functions of the form

\[
V^h(p, y^h) = \log \left( y^h - p_1\beta_1^h - p_2\beta_2^h \right) - \frac{1}{2} \log (p_1p_2)
\]

where \(h = a, b\) and \(\beta_1^h, \beta_2^h\) are parameters. Find the equilibrium price ratio as a function of the property distribution \([R]\).

Outline Answer:
Using Roy’s identity we have, for each \(h\) and each \(i\):

\[
x^h_i = \frac{V^h_i}{V^h_y}
\]

Now we have

\[
V^h_i = -\beta^h_i [y^h - p_1\beta_1^h - p_2\beta_2^h]^{-1} - 1/2p_i,
\]

\[
V^h_y = [y^h - p_1\beta_1^h - p_2\beta_2^h]^{-1}.
\]

Combining the two results we find for each \(h\):

\[
x^h_i = \beta^h_i + \frac{1}{2p_i} \left[ p_1 \left[ R^h_1 - \beta_1^h \right] + p_2 \left[ R^h_2 - \beta_2^h \right] \right]
\]

Defining \(\beta_i = \beta_i^a + \beta_i^b\) and \(R_i = R_i^a + R_i^b\) we obtain the excess demand for good 2:

\[
E_2 = \beta_2 - R_2 + \frac{1}{2p_2} [p_1 [R_1 - \beta_1] + p_2 [R_2 - \beta_2]]
\]

Hence putting \(E_2 = 0\) we get the equilibrium price ratio thus:

\[
\frac{p_1}{p_2} = \frac{R_2 - \beta_2}{R_1 - \beta_1}.
\]
Exercise 7.10 In an economy there are large equal-sized groups of capitalists and workers. Production is organised as in the model of Exercises 2.14 and 6.4. Capitalists’ income consists solely of the profits from the production process; workers’ income comes solely from the sale of labour. Capitalists and workers have the utility functions $x^c_1x^c_2$ and $x^w_1 - [R_3 - x^w_3]^2$ respectively, where $x^h_i$ denotes the consumption of good $i$ by a person of type $h$ and $R_3$ is the stock of commodity 3.

1. If capitalists and workers act as price takers find the optimal demands for the consumption goods by each group, and the optimal supply of labour $R_3 = x^w_3$.

2. Show that the excess demand functions for goods 1,2 can be written as

$$\frac{\Pi}{2p_1} + \frac{1}{2[p_1]^2} - \frac{A}{2p_1}$$

$$\frac{\Pi}{2p_2} - \frac{A}{2p_2}$$

where $\Pi$ is the expression for profits found in Exercise 6.4. Show that in equilibrium $p_1/p_2 = \sqrt{3}$ and hence show that the equilibrium price of good 1 (in terms of good 3) is given by

$$p_1 = \left[\frac{3}{2A}\right]^{1/3}$$

3. What is the ratio of the money incomes of workers and capitalists in equilibrium?

Outline Answer:

1. Given that the capitalist utility function is

$$x^c_1x^c_2$$

it is immediate that in the optimum the capitalists spend an equal share of their income on the two consumption goods and so

$$x^c_i = \frac{\Pi}{2p_i},$$

Worker utility is

$$x^w_1 - [R_3 - x^w_3]^2$$

and the budget constraint is

$$p_1 x^w_i \leq R_3 - x^w_3$$

Maximising (7.18) subject to (7.19) is equivalent to maximising

$$\frac{1}{p_1} [R_3 - x^w_3] - [R_3 - x^w_3]^2,$$

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The FOC is
\[ \frac{1}{p_1} - 2 [R_3 - x_3^w] = 0 \quad (7.21) \]
which gives optimal labour supply as:
\[ R_3 - x_3^w = \frac{1}{2p_1} \quad (7.22) \]
and, from (7.19), the workers’ optimal consumption of good 1 is
\[ x_1^w = \frac{1}{2 [p_1]^2}. \quad (7.23) \]

2. The economy has no stock of good 1 or good 2; workers do not consume good 2; so excess demand for the two goods is, respectively:
\[ x_c^1 + x_w^1 y_1 = \frac{\Pi}{2p_1} + \frac{1}{2 [p_1]^2} - \frac{A}{2} p_1 \quad (7.24) \]
\[ x_2^2 - y_2 = \frac{\Pi}{2p_2} - \frac{A}{2} p_2 \quad (7.25) \]
To find the equilibrium set each of (7.24) and (7.25) equal to zero. This gives
\[ \Pi p_1 + 1 = A [p_1]^3 \quad (7.26) \]
\[ \Pi = A [p_2]^2 \quad (7.27) \]
Substituting in for profits in (7.27) we have
\[ \frac{[p_1]^2 + [p_2]^2}{4} = [p_2]^2 \]
and so
\[ \frac{p_1}{p_2} = \sqrt{3}. \quad (7.28) \]
Substituting for \( \Pi \) and \( p_2 \) we get
\[ p_1 A \left[ \frac{[p_1]^2}{4} + \frac{1}{3} [p_1]^2 \right] + 1 = A [p_1]^3 \]
and, on rearranging, this gives
\[ p_1 = \left[ \frac{3}{2A} \right]^{1/3} \quad (7.29) \]

3. Profits in equilibrium are
\[ A \left[ \frac{1}{4} [p_1]^2 + \frac{1}{3} [p_1]^2 \right] = A \left[ \frac{3}{2A} \right]^{2/3} = \left[ \frac{A}{3} \right]^{1/3} 2^{-2/3}. \]
Given that the price of good 3 is normalised to 1, using (7.22) and (7.29) total labour income in equilibrium is
\[ 1 \cdot \frac{1}{2p_1} = \frac{1}{2} \left[ \frac{3}{2A} \right]^{-1/3} = \left[ \frac{A}{3} \right]^{1/3} 2^{-2/3} \]
So, workers and capitalists get the same money income in equilibrium! Note that this is unaffected by the value of $A$; increases in $A$ could be interpreted as technical progress and so the income distribution remains unchanged by such progress.