Chapter 6

A Simple Economy

Exercise 6.1 *In an economy the activity of digging holes in the ground is carried out by self-employed labourers (single–person firms). The production of one standard-sized hole requires a minimum input of one unit of labour. No self-employed labourer can produce more than one hole.*

1. *Draw the technology set Q for a single firm.*
2. *Draw the technology set Q for two firms.*
3. *Which of the basic production axioms are satisfied by this simple technology?*

Outline Answer

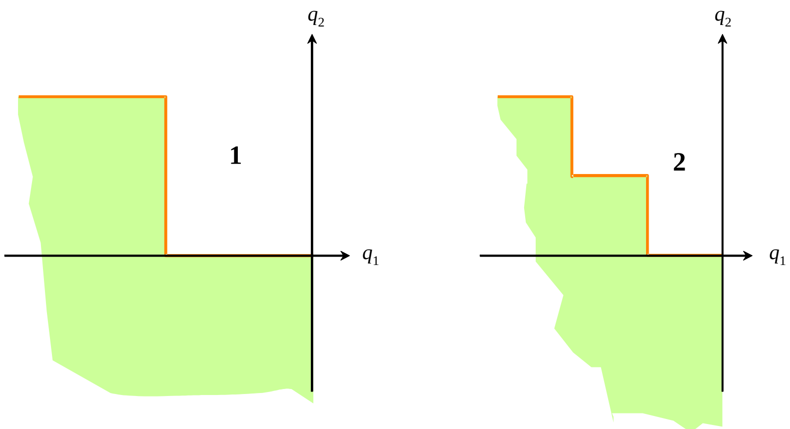

Figure 6.1: Convexification of Production

1. See left-hand panel of Figure 6.1. Good 1 is labour, good 2 is holes.
2. See right-hand panel of Figure 6.1; this has been rescaled to be compatible with the single-firm case..

3. Nonconvexities are present so that the divisibility assumption is violated. Nevertheless the production can be (approximately) convexified by increasing the number of firms.

Exercise 6.2 *Consider the following four examples of technology sets Q:*

$$
\begin{array}{lcl}
A & : & \left\{\mathbf{q} : q_1^2 + q_2^2 + q_3 + q_4 \leq 0;\;\; q_1, q_2 \geq 0; q_3, q_4 \leq 0\right\} \\
B & : & \left\{\mathbf{q} : q_1^\alpha - [-q_2]^\beta - [-q_3]^\gamma \leq 0;\;\; q_1 \geq 0; q_2, q_3 \leq 0\right\} \\
C & : & \left\{\mathbf{q} : \log q_1 - \frac{1}{2}\log(q_2 q_3) \leq 0;\;\; q_1 \geq 0; q_2, q_3 \leq 0\right\} \\
D & : & \left\{\mathbf{q} : q_1 + q_2 + \max\left(q_3, \frac{q_4}{\alpha}\right) \leq 0;\;\; q_1, q_2 \geq 0; q_3, q_4 \leq 0\right\}
\end{array}
$$

1. *Check whether basic production axioms are satisfied in each case.*
2. *Sketch their isoquants and write down the production functions.*
3. *In cases B and C express the production function in terms of the notation used in chapter 2.*
4. *In cases A and D draw the transformation curve.*

Outline Answer:

1. Axioms:

   A: Additivity is not satisfied

   B: All axioms are satisfied if $\alpha = \beta = \gamma < 1$.

   C: All axioms satisfied.

   D: All axioms are satisfied if $\alpha > 0$.

2. Isoquants and production functions:

   A: Isoquants are straight lines.
   $$\Phi(\mathbf{q}) = q_1^2 + q_2^2 + q_3 + q_4$$

   B: If $\alpha = \beta = \gamma < 1$ isoquants will be similar to hyperbolas.
   $$\Phi(\mathbf{q}) = q_1^\alpha - [-q_2]^\beta - [-q_3]^\gamma$$

   C: Similar to B.
   $$\Phi(\mathbf{q}) = \log q_1 - \frac{1}{2}\log(q_2 q_3)$$

   D: Isoquants are rectangular.
   $$\Phi(\mathbf{q}) = q_1 + q_2 + \max\left(q_3, \frac{q_4}{\alpha}\right)$$

3. Single-output production functions. Let $z_2 = -q_2$ and $z_3 = -q_3$. Then:

   B: $q_1 \leq \left[[z_2]^\beta + [z_3]^\gamma\right]^{1/\alpha}$

   C: $q_1 \leq \sqrt{z_2 z_3}$

4. Transformation curve (the boundary of the set Q).

   A: A quarter circle.

   D: A straight line.

Exercise 6.3 *Suppose two identical firms each produce two outputs from a single input. Each firm has exactly 1 unit of input. Suppose that for firm 1 the amounts q_1^1, q_2^2 it produces of the two outputs are given by*

$$\begin{aligned} q_1^1 &= \alpha\theta^1 \\ q_2^1 &= \beta[1-\theta^1] \end{aligned}$$

where θ^1 is the proportion of the input that firm 1 devotes to the production of good 1 and α and β depend on the activity of firm 2 thus

$$\begin{aligned} \alpha &= 1+2\theta^2 \\ \beta &= 1+2[1-\theta^2]. \end{aligned}$$

where θ^2 is the proportion of the input that firm 2 devotes to good 1. Likewise for firm 2:

$$\begin{aligned} q_1^2 &= \alpha'\theta^2 \\ q_2^2 &= \beta'[1-\theta^2] \end{aligned}$$

$$\begin{aligned} \alpha' &= 1+2\theta^1 \\ \beta' &= 1+2[1-\theta^1]. \end{aligned}$$

1. *Draw the production possibility set for firm 1 if firm 2 sets $\theta^2 = \frac{1}{2}$ and firm 2's production possibility set if firm 1 sets $\theta^1 = \frac{1}{2}$.*

2. *Draw the combined production-possibility set.*

Outline Answer:

1. The production possibility set for firm 1 if firm 2 sets $\theta^2 = \dfrac{1}{2}$ are given by the top left-hand panel of Figure 6.2 below. Firm 2's position if firm 1 sets $\theta^1 = \dfrac{1}{2}$ is given in the top right-hand panel. The combined production possibility set is as shown in the lower panel.

2. Let $\theta^2 = \dfrac{1}{2}$; then by definition $\alpha = \beta = 2$, and firm 1's production boundary is given by all the points $(2\theta^1, 2[1-\theta^1])$ where $0 \le \theta^1 \le 1$. A similar argument applies to firm 2. Now consider the sum of their outputs. This will be given by points such as $\theta^1 + \theta^2 + 4\theta^1\theta^2$, $2 - \theta^1 - \theta^2 + 4[1 - \theta^1 - \theta^2][1-\theta^2]$. So, for example, if $\theta^1 = \theta^2 = 0$ we get output $(0, 6)$; if $\theta^1 = \theta^2 = 1$ we get $(6, 0)$ but if $\theta^1 = \theta^2 = \dfrac{1}{2}$ we get $(2, 2)$.

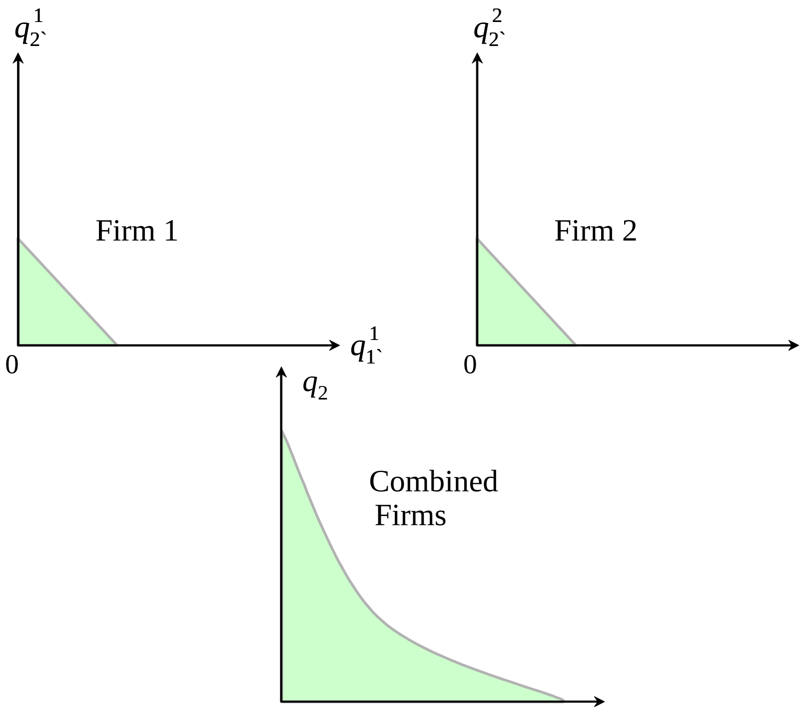

Figure 6.2: Externalities and nonconvexity

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Exercise 6.4 *Take the model of Exercise 2.14. Assuming that production is organised to maximise profits at given prices show that profit-maximising net outputs of goods 1 and 2 are:*

$$\begin{aligned} q_1 &= \frac{A}{2}p_1 \\ q_2 &= \frac{A}{2}p_2 \end{aligned}$$

where p_i is the price of good i expressed in terms of commodity 3, and that maximised profits are

$$\Pi = A\frac{[p_1]^2 + [p_2]^2}{4}.$$

Outline Answer:

Using standard notation for net output profits are given by

$$\Pi = \sum_{i=1}^{3} p_i y_i \tag{6.1}$$

If profits are maximised then production must take place on the transformation curve. Therefore, substituting we get:

$$\Pi = p_1 y_1 + p_2 y_2 - \frac{[y_1]^2 + [y_2]^2}{A} \tag{6.2}$$

To maximise Π just maximise (6.2) with respect to y_1 and y_2. This gives the FOCs:

$$p_i - \frac{2}{A}y_i = 0 \tag{6.3}$$

$i = 1, 2$, and so

$$y_1 = \frac{A}{2}p_1 \tag{6.4}$$

$$y_2 = \frac{A}{2}p_2 \tag{6.5}$$

from which, using (6.2), profits are

$$\Pi = A\frac{[p_1]^2 + [p_2]^2}{4}. \tag{6.6}$$

Exercise 6.5 *Take the model of Exercise 5.3 but suppose that income is exogenously given at y_1 for the first period only. Income in the second period can be obtained by investing an amount z in the first period. Suppose $y_2 = \phi(z)$, where ϕ is a twice-differentiable function with positive first derivative and negative second derivative and $\phi(0) = 0$, and assume that there is a perfect market for lending and borrowing.*

1. *Write down the budget constraint.*
2. *Explain the rôle of the Decentralisation Theorem in this model*
3. *Find the household's optimum and compare it with that of Exercise 5.3.*
4. *Suppose $\phi(z)$ were to be replaced by $\tau\phi(z)$ where $\tau > 1$; how would this affect the solution?*

Outline Answer:

1. The budget constraint is

$$x_1 + \frac{x_2}{1+r} \leq A(z) \tag{6.7}$$

where the present value of lifetime income is

$$A(z) := y_1 - z + \frac{\phi(z)}{1+r} \tag{6.8}$$

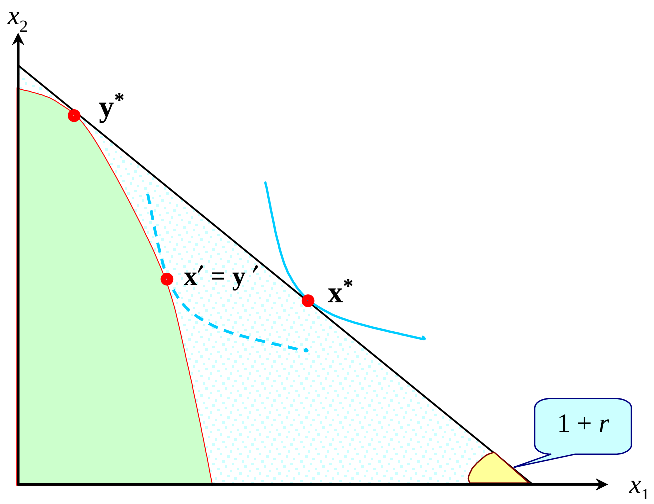

Figure 6.3: Lifetime income: self investment

2. Use the decentralisation result. Given that borrowing/lending is possible at the rate of interest r the right thing to do is to maximise the value of lifetime income ("profits") at point $\mathbf{y}^*$ in Figure 6.3 and then maximise utility using this maximised lifetime income (point $\mathbf{x}^*$).

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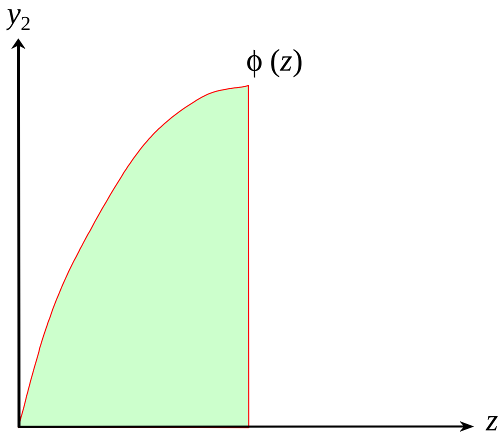

Figure 6.4: The production of human capital

3. Otherwise, without a bank loan one would be constrained to point $\mathbf{x}' = \mathbf{y}' := (y_1, y_2)$. Maximising lifetime income implies

$$\frac{\partial A(z)}{\partial z} = -1 + \frac{\phi_z(z)}{1+r} = 0 \tag{6.9}$$

for an interior solution. This first-order condition implies that the optimal value z^* must satisfy

$$\phi_z(z^*) = 1 + r; \tag{6.10}$$

the marginal rate of transformation of current income into future income equals the price ratio.

1. It is obvious that if ϕ were to be replaced by $\tau\phi$ ($\tau > 1$) the opportunity set is "stretched" in the direction of the vertical axis in Figure 6.4. We expect that maximised lifetime income will increase. So consumption will increase in both periods. What happens to optimal investment? From the first-order condition (6.9) we find

$$\tau\phi_z(z^*) = 1 + r \tag{6.11}$$

from which we could get the optimal investment z^*. If we increase τ then the z^* obtained from (6.11) will change. To see by how much z^* changes differentiate (6.11) thus:

$$\phi_z(z^*) \cdot d\tau + \tau\phi_{zz}(z^*) \cdot dz^* = 0$$

from which we have

$$\frac{dz^*}{d\tau} = -\frac{\phi_z(z^*)}{\tau\phi_{zz}(z^*)} > 0 \tag{6.12}$$

Exercise 6.6 *Apply the model of Exercise 6.5 to an individual's decision to invest in education.*

1. *Assume the parameter τ represents talent. Will more talented people demand purchase more education?*
2. *How is the demand for schooling related to exogenous first-period income y_1?*

Outline Answer:

The optimal purchase of education z^* increases with ability τ, but is independent of initial money income y_1.

Exercise 6.7 *Take the savings model of Exercise 5.4. Suppose now that by investing in education in the first period the consumer can augment his future income. Sacrificing an amount z in period 1 would yield additional income in period 2 of*

$$\tau\left[1 - e^{-z}\right]$$

where $\tau > 0$ is a productivity parameter.

1. *Explain how investment in education is affected by the interest rate. What would happen if the interest rate were higher than $\tau - 1$?*
2. *How is the demand for borrowing affected by (i) an increase in the interest rate r and (ii) an increase in the person's productivity parameter τ?*

Outline Answer

1. By the separation theorem the right thing for the consumer to do is to maximise net income and then choose optimal consumption. By definition net income is
$$M = y_1 - z + \frac{y_2 + \tau\left[1 - e^{-z}\right]}{1+r}$$
The FOC for a maximum is
$$\frac{\partial M}{\partial z} = \frac{\tau e^{-z}}{1+r} - 1 = 0$$
which has as a solution
$$z = z^*(r, \tau) := \log\left(\frac{\tau}{1+r}\right) \tag{6.13}$$
Clearly this only makes sense if
$$\tau \geq 1 + r \tag{6.14}$$
otherwise investment is zero. If (6.14) holds then z^* is decreasing in its first argument and increasing in its second argument.

2. Assuming (6.14) holds, maximised income is:
$$\begin{aligned} M &= M^*(r, \beta) := y_1 - \log\left(\frac{\beta}{1+r}\right) + \frac{y_2 + \beta\left[1 - e^{-z}\right]}{1+r} \\ &= y_1 + \log\left(\frac{1+r}{\beta}\right) - 1 + \frac{y_2 + \beta}{1+r} \end{aligned} \tag{6.15}$$
Hence in this model optimal period-1 consumption is
$$x_1^* = k + \alpha\left[y_1 - k + \log\left(\frac{1+r}{\beta}\right) - 1 + \frac{y_2 - k + \beta}{1+r}\right]. \tag{6.16}$$
Borrowing is given by
$$B^*(r, \beta) := x_1^* + z^*(r, \beta) - y_1$$

and so, on evaluating this from (6.13) and (6.16), we have

$$B^*(r,\beta) = \alpha\left[\frac{y_2 - k + \beta}{1+r} - 1\right] + [1-\alpha]\left[\log\left(\frac{\beta}{1+r}\right) - [y_1 - k]\right]$$

Deifferentiating we have

$$\begin{aligned} \frac{\partial B^*(r,\beta)}{\partial r} &= -\alpha\frac{y_2 - k + \beta}{[1+r]^2} - \frac{1-\alpha}{1+r} \\ \frac{\partial B^*(r,\beta)}{\partial \beta} &= \frac{\alpha}{1+r} + \frac{1-\alpha}{\beta} > 0 \end{aligned}$$

If borrowing decreased with r in the no-education case – see equation (5.17) in Exercise 5.4 – it must certainly decrease in this case too. Borrowing increases with τ.