Exercise 3.1 (The phenomenon of “natural monopoly”) Consider an industry in which all the potential member firms have the same cost function $C$. Suppose it is true that for some level of output $q$ and for any nonnegative outputs $q, q'$ of two such firms such that $q + q' \leq q$, the cost function satisfies the “subadditivity” property

$$C(w, q + q') < C(w, q) + C(w, q').$$

1. Show that this implies that for all integers $N > 1$

$$C(w, q) < NC\left(w, \frac{q}{N}\right), \text{ for } 0 \leq q \leq q.$$

2. What are the implications for the shape of average and marginal cost curves?

3. May one conclude that a monopoly must be more efficient in producing this good?

Outline Answer

1. If $q' = q$ then

$$C(w, 2q) < 2C(w, q).$$

Hence

$$C(w, q) + C(w, 2q) < 3C(w, q).$$

and so, putting $q' = 2q$ we have

$$C(w, 3q) < C(w, q) + C(w, 2q) < 3C(w, q).$$

The result then follows by iteration.

2. If there are economies of scale the average cost of production is decreasing and marginal cost will always be below it. Nevertheless “subadditivity” does not imply economies of scale and therefore we can also observe a standard U-shaped average cost curve.

3. It is cheaper to produce in a single plant rather than using two identical plants. But a monopoly may distort prices.
Exercise 3.2 In a particular industry there are \( n \) profit-maximising firms each producing a single good. The costs for firm \( i \) are

\[
C_0 + cq_i
\]

where \( C_0 \) and \( c \) are parameters and \( q_i \) is the output of firm \( i \). The goods are not regarded as being exactly identical by the consumers and the inverse demand function for firm \( i \) is given by

\[
p_i = \frac{Aq_i^{\alpha-1}}{\sum_{j=1}^{n} q_j^\alpha}
\]

where \( \alpha \) measures the degree of substitutability of the firms’ products, \( 0 < \alpha \leq 1 \).

1. Assuming that each firm takes the output of all the other firms as given, write down the first-order conditions yielding firm 1’s output conditional on the outputs \( q_2, \ldots, q_n \). Hence, using the symmetry of the equilibrium, show that in equilibrium the optimal output for any firm is

\[
q_i^* = \frac{Ao [n - 1]}{n^2c}
\]

and that the elasticity of demand for firm \( i \) is

\[
\frac{n}{n - n\alpha + \alpha}
\]

2. Consider the case \( \alpha = 1 \). What phenomenon does this represent? Show that the equilibrium number of firms in the industry is less than or equal to \( \sqrt{\frac{A}{C_0}} \).

Outline Answer

1. We begin by computing the equilibrium for a typical firm \( i \). Profits for the firm are

\[
\Pi_i = \frac{Aq_i^\alpha}{K} - C_0 - cq_i
\]

where

\[
K := \sum_{j=1}^{n} q_j^\alpha
\]

The first-order condition for maximising (3.1) with respect to \( q_i \) (taking all the other \( q_j \) as given) is

\[
\frac{\partial \Pi_i}{\partial q_i} = \frac{Aq_i^\alpha - 1}{K} - \frac{Aq_i^{2\alpha - 1}}{K^2} - c = 0
\]

If all firms are identical, then in equilibrium all firms must produce the same amount and so

\[
K = nq_i^{*\alpha}
\]

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Substituting (3.3) in (3.2) we get

$$\frac{A\alpha}{n} - \frac{A\alpha}{n^2} - cq_i^* = 0$$

(3.4)

from which the result follows immediately. To find the elasticity of demand for firm $i$ take logs of the inverse demand curve (in the question) and differentiate with respect to $q_i$

$$-\frac{q_i}{p_i} \frac{\partial p_i}{\partial q_i} = 1 - \alpha + \frac{\alpha q_i^2}{K}$$

(3.5)

To find the elasticity in the neighbourhood of the equilibrium substitute (3.3) in (3.5) and take the reciprocal.

2. The case $\alpha = 1$ represents a situation where the goods are perfect substitutes. We then find that firm $i$’s profits are

$$\Pi_i^* = \frac{Aq_i^*}{K} - C_0 - cq_i^*$$

(3.6)

$$= \frac{A}{n} - C_0 - \frac{A[n - 1]}{n^2}$$

$$= \frac{A}{n^2} - C_0$$

(3.7)

Requiring that the right-hand side of (3.7) be non-negative implies

$$n \leq \sqrt{\frac{A}{C_0}}$$

(3.8)
Exercise 3.3 A firm has the cost function

\[ F_0 + \frac{1}{2} a q_i^2 \]

where \( q_i \) is the output of a single homogenous good and \( F_0 \) and \( a \) are positive numbers.

1. Find the firm’s supply relationship between output and price \( p \); explain carefully what happens at the minimum-average-cost point \( p := \sqrt{2aF_0} \).

2. In a market of a thousand consumers the demand curve for the commodity is given by

\[ p = A - bq \]

where \( q \) is total quantity demanded and \( A \) and \( b \) are positive parameters.

If the market is served by a single price-taking firm with the cost structure in part 1 explain why there is a unique equilibrium if \( b \leq a \left( \frac{A}{p} - 1 \right) \) and no equilibrium otherwise.

3. Now assume that there is a large number \( N \) of firms, each with the above cost function: find the relationship between average supply by the \( N \) firms and price and compare the answer with that of part 1. What happens as \( N \to \infty \)?

4. Assume that the size of the market is also increased by a factor \( N \) but that the demand per thousand consumers remains as in part 2 above. Show that as \( N \) gets large there will be a determinate market equilibrium price and output level.

Outline Answer

1. Given the cost function

\[ F_0 + \frac{1}{2} a q_i^2 \]

marginal cost is \( a q_i \) and average cost is \( F_0/q_i + \frac{1}{2} a q_i \). Marginal cost intersects average cost where

\[ a q_i = F_0/q_i + \frac{1}{2} a q_i \]

i.e. where output is

\[ q := \sqrt{2F_0/a} \]  

and marginal cost is

\[ p := \sqrt{2aF_0} \]

For \( p > p \) the supply curve is identical to the marginal cost curve \( q_i = p/a \); for \( p < p \) the firm supplies 0 to the market; at \( p = p \) the firm supplies either 0 or \( q \). There is no price which will induce a supply in the interior of the interval \((0, q)\). Summarising, firm \( i \)'s optimal output is given by

\[ q_i^* = S(p) := \begin{cases} 
  p/a, & \text{if } p > p \\
  q \in \{0, q\}, & \text{if } p = p \\
  0, & \text{if } p < p 
\end{cases} \]

(3.11)
2. The equilibrium, if it exists, is found where supply = demand at a given price. This would imply
\[
\frac{p}{a} = \frac{A - p}{b} \\
p = \frac{aA}{a + b}
\]
which would, in turn, imply an equilibrium quantity
\[
q = \frac{A}{a + b}
\]
but it can only be valid if \(\frac{A}{a + b} \geq \frac{q}{a}\). Noting that \(q = p/a\) this condition is equivalent to \(\frac{A}{a + b} - 1 \geq b\).

3. If there are \(N\) such firms, each firm responds to price as in (3.11), and so the average output \(\bar{q} := \frac{1}{N} \sum_{i=1}^{N} q_i^*\) is given by
\[
\bar{q} = \begin{cases} 
  p/a, & \text{if } p > p^0 \\
  q \in J(q) & \text{if } p = p^0 \\
  0, & \text{if } p < p^0 
\end{cases}
\] (3.12)
where \(J(q) := \{\frac{i}{N}q : i = 0, 1, ..., N\}\). As \(N \to \infty\) the set \(J(q)\) becomes dense in \([0, q]\), and so we have the average supply relationship:
\[
\bar{q} = \begin{cases} 
  p/a, & \text{if } p > p^0 \\
  q \in [0, q] & \text{if } p = p^0 \\
  0, & \text{if } p < p^0 
\end{cases}
\] (3.13)

4. Given that in the limit the average supply curve is continuous and of the piecewise linear form (3.13), and that the demand curve is a downward-sloping straight line, there must be a unique market equilibrium. The equilibrium will be found at \(\left(\bar{p}, \frac{A - p}{b}\right)\) which, using (3.10) is \(\left(\sqrt{2ap_0}, \frac{A - \sqrt{2ap_0}}{b}\right)\). Using (3.9) this can be written \(\left(\bar{p}, \beta q\right)\) where
\[
\beta := \frac{A - p}{bp/a}
\]
In the equilibrium a proportion \(\beta\) of the firms produce \(q\) and \(1 - \beta\) of the firms produce 0.
Exercise 3.4  A firm has a fixed cost $F_0$ and marginal costs

$$c = a + bq$$

where $q$ is output.

1. If the firm were a price-taker, what is the lowest price at which it would be prepared to produce a positive amount of output? If the competitive price were above this level, find the amount of output $q^*$ that the firm would produce.

2. If the firm is actually a monopolist and the inverse demand function is

$$p = A - \frac{1}{2}Bq$$

(where $A > a$ and $B > 0$) find the expression for the firm’s marginal revenue in terms of output. Illustrate the optimum in a diagram and show that the firm will produce

$$q^* := \frac{A - a}{b + B}$$

What is the price charged $p^*$ and the marginal cost $c^*$ at this output level? Compare $q^*$ and $q^*$.

3. The government decides to regulate the monopoly. The regulator has the power to control the price by setting a ceiling $p_{\text{max}}$. Plot the average and marginal revenue curves that would then face the monopolist. Use these to show:

(a) If $p_{\text{max}} > p^*$ the firm’s output and price remain unchanged at $q^*$ and $p^*$

(b) If $p_{\text{max}} < c^*$ the firm’s output will fall below $q^*$.

(c) Otherwise output will rise above $q^*$.

Outline Answer

1. Total costs are

$$F_0 + aq + \frac{1}{2}bq^2$$

So average costs are

$$\frac{F_0}{q} + a + \frac{1}{2}bq$$

which are a minimum at

$$q = \sqrt{\frac{2F_0}{b}}$$

(3.14)

where average costs are

$$\sqrt{2bF_0} + a$$

(3.15)

Marginal and average costs are illustrated in Figure 3.1 notice that MC is linear and that AC has the typical U-shape if $F_0 > 0$. For a price above the level (3.15) the first-order condition for maximum profits is given by

$$p = a + bq$$

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Figure 3.1: Perfect competition

from which we find

\[ q^* := \frac{p - a}{b} \]

– see figure 3.1.

2. If the firm is a monopolist marginal revenue is

\[ \frac{\partial}{\partial q} \left[ Aq - \frac{1}{2} Bq^2 \right] = A - Bq \]

Hence the first-order condition for the monopolist is

\[ A - Bq = a + bq \] (3.16)

from which the solution \( q^{**} \) follows. Substituting for \( q^{**} \) we also get

\[ c^{**} = A - Bq^{**} = \frac{Ab + Ba}{B + b} \] (3.17)

\[ p^{**} = A - \frac{1}{2} Bq^{**} = c^{**} + \frac{1}{2} B A - \frac{a}{b + B} \] (3.18)

– see figure 3.2.

3. Consider how the introduction of a price ceiling will affect average revenue. Clearly we now have

\[ AR(q) = \begin{cases} p_{\max} & \text{if } q \leq q_0 \\ A - \frac{1}{2} Bq & \text{if } q \geq q_0 \end{cases} \] (3.19)

where \( q_0 := 2 [A - p_{\max}] / B \): average revenue is a continuous function of \( q \) but has a kink at \( q_0 \). From this we may derive marginal revenue which is

\[ MR(q) = \begin{cases} p_{\max} & \text{if } q < q_0 \\ A - Bq & \text{if } q > q_0 \end{cases} \] (3.20)

– notice that there is a discontinuity exactly at \( q_0 \). The modified curves (3.19) and (3.20) are shown in Figure 3.3 notice that they coincide in
the flat section to the left of \( q_0 \). Clearly the outcome depends crucially on whether MC intersects (modified) MR (a) to the left of \( q_0 \), (b) to the right of \( q_0 \), (c) in the discontinuity exactly at \( q_0 \). Case (c) is illustrated, and it is clear that output will have risen from \( q^{**} \) to \( q_0 \). The other cases can easily be found by appropriately shifting the curves on Figure 3.3.
Exercise 3.5  A monopolist has the cost function

\[ C(q) = 100 + 6q + \frac{1}{2}q^2 \]

1. If the demand function is given by

\[ q = 24 - \frac{1}{4}p \]

calculate the output-price combination which maximises profits.

2. Assume that it becomes possible to sell in a separate second market with demand determined by

\[ q = 84 - \frac{3}{4}p. \]

Calculate the prices which will be set in the two markets and the change in total output and profits from case 1.

3. Now suppose that the firm still has access to both markets, but is prevented from discriminating between them. What will be the result?

Outline Answer

1. Maximizing the simple monopolist’s profits

\[ \Pi_0 = (96 - 4q)q - \left(100 + 6q + \frac{q^2}{2}\right) \]

with respect to \( q \) yields optimum output of \( q_0 = 10 \). Hence \( p_0 = 56 \) and \( \Pi_0 = 350 \).

2. Now let the monopolist sell \( q_1 \) in market 1 for price \( p_1 \) and \( q_2 \) in market 2 for price \( p_2 \). The new problem is to choose \( q_1, q_2 \) so as to maximise the function

\[ \Pi_{12} = (96 - 4q_1)q_1 + (112 - \frac{4}{3}q_2)q_2 - \left(100 + 6q_1 + 6q_2 + \frac{(q_1 + q_2)^2}{2}\right). \]

First-order conditions yield

\[ 9q_1 + q_2 = 90 \]
\[ q_1 + \frac{11}{3}q_2 = 106. \]

Solving we find \( q_1 = 7, q_2 = 27 \) and hence \( p_1 = 68, p_2 = 76 \) and \( \Pi_{12} = 1646 \).

3. If we abandon discrimination, a uniform price \( \hat{p} \) must be charged. If \( \hat{p} > 112 \) nothing is sold to either market. If \( 112 > \hat{p} > 96 \) only market 2 is served. If \( 96 > \hat{p} \) both market are served and the demand curve is \( \hat{q} = 108 - \hat{p} \). Clearly this is the relevant region. Maximising simple monopoly profits we find \( \hat{q} = 34, \hat{p} = 74 \) and \( \hat{\Pi} = 1634 \).

Hence the total output is identical to that under discrimination, \( p_1 < \hat{p} < p_2 \) and \( \Pi_{12} > \hat{\Pi} \). These results are quite general.