Questions

1. A competitive firm produces a single output from three inputs; production is given by $q \leq [z_1 z_2 z_3]^{\frac{1}{3}}$ where q is the quantity of output and z_1, z_2, z_3 are quantities of the inputs.

   (a) If the input prices are given by w_1, w_2, w_3 show that the cost function is $3q\,[w_1 w_2 w_3]^{\frac{1}{3}}$.[3 marks]

   (b) Sketch the (long-run) MC and AC curves. [2 marks]

   (c) Input 3 is fixed in the short run. Using the diagram from part (b) explain what the short-run MC and AC must look like. [3 marks]

2.

   (a) In economy A goods 1 and 2 can be produced using a single good 3 as an input according to the following constraint

   $$a_{13}q_1 + a_{23}q_2 \leq R_3$$

   where a_{13}, a_{23} are positive constants, q_1, q_2 are the outputs of the two goods and R_3 is the given resource stock of good 3. Draw the production possibility set for goods 1 and 2 in economy A. [2 marks]

   (b) In economy B production of goods 1 and 2 requires the use of goods 3,4 and 5 as inputs according to the following three constraints (which must all hold):

   $$a_{1j}q_1 + a_{2j}q_2 \leq R_j,\ j = 3, 4, 5,$$

   where a_{1j}, a_{2j} are positive constants and R_j is the given resource stock of good j. Draw the production possibility set for goods 1 and 2 in economy B: explain why it must be convex. [3 marks]

   (c) Explain what would happen to the production sets of economy A and economy B if additional stocks of good 3 became available.[3 marks]

3. A person has the utility function $U(x_1, x_2) = 2\sqrt{x_1} + x_2$. Write the price of good 1 as p, assume that the price of good 2 is fixed at 1 and assume the income y is sufficiently high to ensure that positive amounts of both goods are bought.

   (a) If the initial value of p is also 1 show that the compensating variation of a price rise Δp is $-\Delta p\,[1 + \Delta p]^{-1}$. [4 marks]

   (b) Again assume that p initially has the value 1. The government now imposes a tax t on good 1 in order to raise revenue. Show that the loss to a consumer exceeds the revenue raised by the government by an amount $t^2\,[1 + t]^{-2}$. [4 marks]

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7. A person lives for $T+1$ years and consumes a quantity x_{it} of good i in period t, where $i = 1, 2, ..., n$ and $t = 0, 1, ..., T$.

   (a) His utility is given by

   $$\sum_{t=0}^{T}\sum_{i=1}^{n}\alpha_i\beta_t \log\left(x_{it}-\gamma_{it}\right)$$

   where the $\alpha_i, \beta_t, \gamma_{it}$ are non-negative parameters such that $\sum_{i=1}^{n}\alpha_i = \sum_{t=0}^{T}\beta_t = 1$ and $\beta_t/\beta_{t-1} = \delta, t = 1, ..., T$, where $0 < \delta < 1$.

      i. Sketch the indifference curves for goods i and j at a given time t; interpret the slope. [1 mark]

      ii. Interpret the parameters γ_{it}. [1 mark]

      iii. Sketch the indifference curves for good i consumed in period t and the same good i consumed in period $t-1$; interpret the slope. [1 mark]

   (b) The person receives an exogenous income y_t in period t; the price p_{it} of good i in period t is known in advance; it is also known that there is a per-period interest rate r that is constant through time. If there are no other restrictions on borrowing and lending what is the person's lifetime budget constraint? [4 marks]

   (c) Find the quantities x_{it}^* that maximise the utility function in part (a) subject to the lifetime budget constraint in part (b). [5 marks]

   (d) Find the effect of an increase in the price p_{it} on the demand for good $j \neq i$ at time $t' \neq t$. Interpret your answer. [4 marks]

   (e) Suppose that for commodity i the parameter γ_{it} is the same for all t. If p_{it} is known to be the same for all t will the optimal consumption of good i be the same for all t? Explain your answer. [4 marks]

8. In a two-commodity exchange economy there are two types of people. People of type a own all of commodity 1 (100 units per person) and people of type b own all of commodity 2 (200 units per person). Let x_i^h denote the consumption of good i by a person in group h, $i = 1, 2$, $h = a, b$.

   (a) Assume that preferences of type a are given by $\log x_1^a + \log x_2^a$. Show that in a competitive market each person's demand for the two commodities will be $\mathbf{x}^a = (50, 50p_1/p_2)$. Find the offer curve for a person of type a. [5 marks]

   (b) Assume that preferences of a type-b person are given by $3\log x_1^b + \log x_2^b$. Find a type-b person's demand for the two goods and the offer curve. If there are equal numbers of people of each type use the result from part (a) to find the competitive-equilibrium price and allocation. [5 marks]

(c) Repeat part (b) for the case where type-b preferences are $3x_1^b + x_2^b$. [5 marks]

(d) Repeat part (b) for the case where type-b preferences are $\min\left\{x_1^b, x_2^b\right\}$. [5 marks]

9. Anne has an initial stock of wealth W and risks losing some of this wealth through fire. The probability of such a fire is known to be π and the loss if the fire occurs would be L (where $L < W$). Insurance cover against a fire is available at a premium κ, where $\kappa > \pi L$; it is also possible to take out partial cover on a pro-rata basis, so that an amount tL of the loss can be covered at cost $t\kappa$ where $0 < t < 1$.

   (a) Draw and explain a diagram that depicts Anne's budget set. [4 marks]

   (b) Anne's preferences under uncertainty are given by a standard von Neumann-Morgenstern utility function. Explain why Anne will not choose full insurance, even if she is risk averse. [5 marks]

   (c) Assuming that she is risk averse, find the conditions that will determine Anne's optimal value of t. [6 marks]

   (d) Beth's wealth is greater than Anne's, but she faces the same possible loss through fire L with the same probability π; she can get insurance cover on exactly the same terms as Anne. Beth has the same preferences as Anne and these preferences exhibit decreasing absolute risk aversion. Use your answer to part (c) to show that the insurance cover Beth chooses is less than that chosen by Anne. [5 marks]

Outline Answers

1.

   (a) The problem can be written as

$$\min w_1 z_1 + w_2 z_2 + w_2 z_3 \text{ subject to } 3\log q \leq \log(z_1) + \log(z_2) + \log(z_3).$$

with associated Lagrangian

$$w_1 z_1 + w_2 z_2 + w_2 z_3 + \lambda\left[3\log q - \log(z_1) - \log(z_2) - \log(z_3)\right].$$

The first-order conditions for an interior optimum are

$$w_i - \frac{\lambda^*}{z_i^*} = 0, i = 1, ..., 3 \quad (1)$$

$$\log(z_1^*) + \log(z_2^*) + \log(z_3^*) = 3\log q. \quad (2)$$

Substituting from (1) into (2) we get

$$\log\left(\frac{\lambda^*}{w_1}\right) + \log\left(\frac{\lambda^*}{w_2}\right) + \log\left(\frac{\lambda^*}{w_3}\right) = 3\log q,$$

$$3\log\lambda^* = 3\log q + \log(w_1 w_2 w_3),$$

so that

$$\lambda^* = q\,[w_1 w_2 w_3]^{\frac{1}{3}}\,. \tag{3}$$

Using (1) and (3) we see that

$$\begin{aligned} w_1 z_1^* + w_2 z_2^* + w_3 z_3^* &= 3\lambda^* \\ &= 3q\,[w_1 w_2 w_3]^{\frac{1}{3}}\,. \end{aligned} \tag{4}$$

This is the cost function $C(\mathbf{w}, q)$.

(b) Given that $C(\mathbf{w}, q)$ is linear in q it is clear that MC and AC must each be a horizontal straight line at $3\,[w_1 w_2 w_3]^{\frac{1}{3}}$.

(c) If input 3 is fixed at $\bar{z}$, then in the short-run we have $q \leq a\,[z_1 z_2]^{\frac{1}{3}}$, where $a := [\bar{z}]^{1/3}$. It is clear that in the short-run we have decreasing returns everywhere and for the answer it is sufficient to point out that this will imply everywhere increasing MC and (taking into account the fixed cost of input 3) a U-shaped average cost curve which touches the long-run AC at its lowest poipnt. To confirm this (although this is not necessary for the answer) we note that, from part (a), in the short run, the Lagrangian is

$$w_1 z_1 + w_2 z_2 + \lambda\,[3\log q - \log(z_1) - \log(z_2) - \log(\bar{z})]\,.$$

and the FOC are now

$$w_i - \frac{\lambda^{**}}{z_i^{**}} = 0, i = 1, 2 \tag{5}$$

$$\log(z_1^{**}) + \log(z_2^{**}) = 3\log(q/a)\,. \tag{6}$$

From this we obtain

$$\log(z_i^{**}) = \log(\lambda^{**}) - \log(w_i) \quad , i = 1, 2 \tag{7}$$

$$\log(z_1^{**}) + \log(z_2^{**}) = 2\log(\lambda^{**}) - \log(w_1) - \log(w_2) \tag{8}$$

$$\log(\lambda^{**}) = \frac{3}{2}\log\left(\frac{q}{a}\right) + \frac{1}{2}\log(w_1 w_2) \tag{9}$$

$$\log(w_i z_i^{**}) = \frac{3}{2}\log\left(\frac{q}{a}\right) + \frac{1}{2}\log(w_1 w_2)\,, i = 1, 2 \tag{10}$$

So short-run variable costs are given by

$$w_1 z_1^{**} + w_2 z_2^{**} = 2\left[\frac{q}{a}\right]^{3/2} [w_1 w_2]^{1/2} = 2q^{3/2}\left[\frac{w_1 w_2}{\bar{z}}\right]^{1/2}$$

Add the fixed costs to this to get total short-run costs:

$$w_1 z_1^{**} + w_2 z_2^{**} + w_3 \bar{z} = 2q^{3/2}\left[\frac{w_1 w_2}{\bar{z}}\right]^{1/2} + w_3 \bar{z} \tag{11}$$

Dividing (11) by q we see that short-run AC is U-shaped and differentiating (11) with respect to q we see that, for $q > 0$, MC is positive and increasing. At the point where $\bar{z} = q\left[w_1 w_2\right]^{1/3} w_3^{-2/3}$ (the optimal value in the long run), the right-hand side of equation (11) becomes

$$2q^{3/2}\left[\frac{w_1 w_2}{q\left[w_1 w_2\right]^{1/3}}\right]^{1/2} w_3^{1/3} + q\left[w_1 w_2\right]^{1/3} w_3^{1/3} = 3q\left[w_1 w_2 w_3\right]^{1/3}.$$

So, dividing this by q, we can see that, exactly at that point, the short-run AC equals the long-run AC.

2.

(a) The boundary of the possibility set is given by the line segment consisting of points (q_1, q_2) satisfying $a_{13}q_1 + a_{23}q_2 = R_3$ and lying between the two corners $(R_3/a_{13}, 0)$ and $(0, R_3/a_{23})$; the whole set consists of the triangle formed by these two corners and the origin and the interior of the triangle.

(b) Each constraint restricts you to a triangular shape, a convex set. So the attainable set is the intersection of three such convex sets, which must also be convex. An example is the polygon shown in Figure 1 [without further restriction on the parameters other shapes are possible].

(c) In economy A the boundary moves out, parallel to itself. In economy B the boundary line of "points satisfying constraint 3" moves outwards. Small change in this boundary leaves the general shape of the production-possibility set unchanged; a large change may alter the number of vertices of the polygon.

3. Assuming that the budget constraint is binding we have $px_1 + x_2 = y$.

(a) Substitute for x_2 into the given utility function and we find utility can be written as

$$U(x_1, x_2) = 2\sqrt{x_1} + y - px_1$$

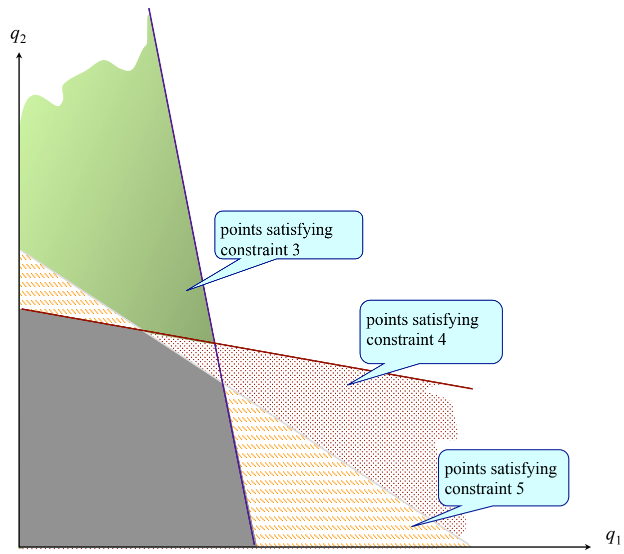
from which it is clear that utility is maximised where $x_1^{-1/2} - p = 0$ so that the demand for good 1 is $x_1 = p^{-2}$ and for good 2 is $y - p^{-1}$ (the assumption in the question assures us that we have an interior solution).

(b) Substituting the optimised value for $x_1$into the utility function, we see that the indirect utility function is therefore

$$2\sqrt{p^{-2}} + y - p^{-1} = \frac{1}{p} + y$$

If p rises from 1 to $1 + \Delta p$ then utility is initially $1 + y$ and after the price rise it is $\frac{1}{1+\Delta p} + y$; so the CV of the price increase is$[1 + \Delta p]^{-1} -$

Figure 1: The attainable set



$1 = -\Delta p\,[1+\Delta p]^{-1} < 0$ (which, in this case of a zero-income-effect utility function, is also the EV).

(c) If $p = 1$ and then rises to $1+t$ then demand at the tax-inclusive price is $[1+t]^{-2}$ and the government raises through the tax an amount $t\,[1+t]^{-2}$. Using the result of part (b) the CV of the price increase is $-t\,[1+t]^{-1}$. Adding the gain to the government and the loss to the consumer we have

$$t\,[1+t]^{-2} - t\,[1+t]^{-1} = -t^2\,[1+t]^{-2}$$

which is clearly negative.

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7.

(a) (i) The indifference curves are all "shifted" Cobb-Douglas with asymptotes γ_{it} and γ_{jt} . If x_j is on the horizontal axis and $x_i$on the vertical then the slope is the marginal rate of substitution of good i for good

j (at time t). (ii) γ_{it} represents the "minimum necessary" consumption of good i in period t. (iii) Again the indifference curves are all "shifted" Cobb-Douglas, this time with asymptotes γ_{it} and $\gamma_{i,t-1}$. If consumption in period t is on the horizontal axis and consumption in period $t-1$ is on the vertical axis then the slope is the marginal rate of substitution of good i in $t-1$ for good i in period t. If the same amount of good i is consumed both periods then this MRS equals δ, a parameter reflecting pure time preference.

(b) If the current price of good i in period t is $p_{it}$and the quantity consumed of good i in period t is x_{it} then, using the assumption of a constant interest rate, total lifetime expenditure, discounted to period 0, is

$$\sum_{t=0}^{T} \frac{1}{[1+r]^t} \sum_{i=1}^{n} p_{it}x_{it} = \sum_{t=0}^{T} \sum_{i=1}^{n} p'_{it}x_{it},$$

where

$$p'_{it} := p_{it} [1+r]^{-t} \tag{12}$$

is the price of good i in period t discounted back to period 0. Likewise, total lifetime income discounted to period 0 is

$$M := \sum_{t=0}^{T} \frac{y_t}{[1+r]^t}$$

So, if people can borrow or lend freely and all future prices are known, the relevant budget constraint is

$$\sum_{t=0}^{T} \sum_{i=1}^{n} p'_{it}x_{it} \leq M$$

(c) The Langragian is

$$\sum_{t=0}^{T} \sum_{i=1}^{n} \alpha_i \beta_t \log(x_{it} - \gamma_{it}) + \mu \left[M - \sum_{t=0}^{T} \sum_{i=1}^{n} p'_{it}x_{it} \right].$$

The $n[T+1]+1$ FOCs for an interior solution are

$$\frac{\alpha_i \beta_t}{x^*_{it} - \gamma_{it}} - \mu^* p'_{it} = 0, \; i = 1, ..., n; t = 0, ..., T. \tag{13}$$

$$\sum_{t=0}^{T} \sum_{i=1}^{n} p'_{it}x^*_{it} = M \tag{14}$$

From (13) we have

$$x^*_{it} = \gamma_{it} + \frac{\alpha_i \beta_t}{\mu^* p'_{it}}, \; i = 1, ..., n; t = 0, ..., T. \tag{15}$$

Using the expressions (15) in(14) we find

$$M_0 + \frac{1}{\mu^*}\sum_{t=0}^{T}\sum_{i=1}^{n}\alpha_i\beta_t = M \tag{16}$$

where

$$M_0 := \sum_{t=0}^{T}\sum_{i=1}^{n}p'_{it}\gamma_{it} \tag{17}$$

is the cost of all the minimum necessary consumption baskets over the lifetime, discounted back to period 0. The adding-up properties of $\alpha_i\beta_t$ (given in the question) and (16) jointly imply

$$\mu^* = \frac{1}{M - M_0};$$

using this in (16) we find

$$x^*_{it} = \gamma_{it} + \alpha_i\beta_t\frac{M - M_0}{p'_{it}},\ i = 1, ..., n; t = 0, ..., T. \tag{18}$$

(d) From (18) it is clear that the only way that the price of good i at time t can affect the demand for good $j \neq i$ at time $t' \neq t$ is through the cost of minimum consumptions M_0 defined in (17). A rise in this price will increase M_0 (and thus reduce $x^*_{jt'}$ if and only if $\gamma_{it} > 0$.

(e) Under the stated conditions (18) yields

$$x^*_{it} = \gamma_i + \alpha_i\beta_t\frac{M - M_0}{p'_{it}},$$

$$x^*_{it-1} = \gamma_i + \alpha_i\beta_{t-1}\frac{M - M_0}{p'_{it-1}},$$

so $x^*_{it-1} = x^*_{it}$ if and only if $\beta_t/\beta_{t-1} = p'_{it}/p'_{it-1}$. We know from the question that $\beta_t = \delta\beta_{t-1}$ and, from (12) we see that

$$\frac{p'_{it}}{p'_{it-1}} = \frac{1}{1+r}\cdot\frac{p_{it}}{p_{it-1}}$$

So, under the stated conditions, if p_{it} stays constant for all t then $x^*_{it-1} = x^*_{it}$ for all t if and only if $\delta = 1/[1 + r]$. This is equivalent to saying that, under the stated conditions, the person chooses a level consumption profile if and only if his per-period personal discount factor (the pure time preference parameter) δ is the same as the per-period market discount factor $1/[1 + r]$.

8.

(a) The income of a type-a person is 100ρ where $\rho := p_1/p_2$ and we are using good 2 as *numéraire*. The Lagrangian is

$$\log x_1^a + \log x_2^a + \lambda\left[100\rho - \rho x_1^a - x_2^a\right] \tag{19}$$

First order conditions for an interior maximum (19) of are

$$\begin{aligned} \frac{1}{x_1^a} - \lambda\rho &= 0 \\ \frac{1}{x_2^a} - \lambda &= 0 \\ 100\rho - \rho x_1^a - x_2^a &= 0 \end{aligned}$$

Solving these we find $\lambda = \frac{1}{50\rho}$ and so the demands will be

$$\mathbf{x}^a = \begin{bmatrix} 50 \\ 50\rho \end{bmatrix} \tag{20}$$

and the offer curve will simply be a vertical straight line at $x_1^a = 50$.

(b) The income of a type-b person is 200. They have Cobb-Douglas preferences similar to those in part (a). Use a small modification of the same analysis to get their demands:

$$\mathbf{x}^b = \begin{bmatrix} \frac{3}{4} \times \frac{200}{\rho} \\ \frac{1}{4} \times 200 \end{bmatrix} = \begin{bmatrix} \frac{150}{\rho} \\ 50 \end{bmatrix} \tag{21}$$

and their offer curve will simply be a horizontal straight line at $x_2^b = 50$. Equilibrium is where the a-offer curve intersects the b-offer curve. Given (20), (21) and the fact that there are 100 units per person of commodity 1 and 200 units per person of commodity 2, the materials balance condition then means that the equilibrium allocation must be

$$\begin{aligned} \mathbf{x}^a &= \begin{bmatrix} 50 \\ 150 \end{bmatrix} \\ \mathbf{x}^b &= \begin{bmatrix} 50 \\ 50 \end{bmatrix} \end{aligned}$$

and – from (20) or (21) – the equilibrium price is $\rho = 3$.

(c) If type-b persons have linear preferences $3x_1^b + x_2^b$. their demands will be

$$\begin{aligned} \mathbf{x}^b &= \mathbf{x}', \text{ if } \rho > 3 \\ \mathbf{x}^b &\in [\mathbf{x}', \mathbf{x}''], \text{ if } \rho = 3 \\ \mathbf{x}^b &= (200/\rho, 0), \text{ if } \rho < 3 \end{aligned}$$

where $\mathbf{x}' := (0, 200)$, $\mathbf{x}'' := (200/3, 0)$, and their offer curve will consist of the union of the line segment $[\mathbf{x}', \mathbf{x}'']$ and the line segment from $\mathbf{x}''$ to $(\infty, 0)$. Again from (20) we have $x_1^a = 50$, and so $x_1^b = 50$, by materials balance. Using the fact that the equilibrium must lie on the type-b offer curve we see that the solution must lie on the straight line from $(0, 200)$ to $(200/3, 0)$ we find that $x_2^b = 50$ and, from the materials balance condition $x_2^a = 200 - 50 = 150$ (as in the previous case). By the same reasoning as in the previous case the equilibrium price must be $\rho = 3$.

(d) If type-b persons have Leontief preferences $\min\left\{x_1^b, x_2^b\right\}$ then they always consume so that $x_1^b = x_2^b$; in view of the budget constraint

$$\rho x_1^b + x_2^b = 200$$

their demands will be

$$\mathbf{x}^b = \begin{bmatrix} \frac{200}{\rho+1} \\ \frac{200}{\rho+1} \end{bmatrix}$$

and their offer curve is just the straight line $x_2^b = x_1^b$. Using the materials-balance condition we again find $x_2^a = 200 - 50 = 150$ (as in the previous cases). As before the equilibrium price must be 3.

9.

(a) Standard state-space diagram for a two-state model (NO LOSS,LOSS). Consider Anne's ex-post wealth using the two-state model (NO LOSS,LOSS). If Anne remained uninsured it would be

$$(W, W - L)\,; \tag{22}$$

If she insures fully it is

$$(W - \kappa, W - \kappa)\,. \tag{23}$$

So if she insures a proportion t, wealth in the two states will be $([1-t]\,W + t\,[W-\kappa]\,, [1-t]\,[W-L] + t\,[W-\kappa])$ which becomes

$$(W - t\kappa, W - t\kappa - [1-t]\,L) \tag{24}$$

Diagram should illustrate points (22) and (23)and the points such as (24) on the straight line joining them. Explain why over-insurance is unlikely to be possible

(b) Expected utility is given by

$$\mathcal{E}u = [1-\pi]\,u\,(W - t\kappa) + \pi u\,(W - t\kappa - [1-t]\,L)$$

Therefore

$$\frac{\partial \mathcal{E}u}{\partial t} = -\,[1-\pi]\,\kappa u'\,(W - t\kappa) + [L-\kappa]\,\pi u'\,(W - t\kappa - [1-t]\,L)$$

where u' denotes the first derivative of u. Consider what happens in the neighbourhood of $t = 1$ (full insurance). We get

$$\begin{aligned} \left.\frac{\partial \mathcal{E}u}{\partial t}\right|_{t=1} &= -[1-\pi]\,\kappa u'(W-\kappa) + [L-\kappa]\,\pi u'(W-\kappa) \\ &= [L\pi - \kappa]\,u'(W-\kappa) \end{aligned} \tag{25}$$

We know that $u'(W-\kappa) > 0$ (positive marginal utility of wealth) and, by assumption, $L\pi < \kappa$. Therefore (25) is strictly negative which means that in the neighbourhood of full insurance ($t = 1$) Anne could increase expected utility by cutting down on the insurance cover.

(c) For an interior maximum we have

$$\frac{\partial \mathcal{E}u}{\partial t} = 0$$

which means that the optimal value is given as the solution t^* to the equation

$$-[1-\pi]\,\kappa u'(W-t\kappa) + [L-\kappa]\,\pi u'(W-t\kappa-[1-t]\,L) = 0 \tag{26}$$

(d) Since Beth has the same preferences as Anne we can figure out what happens just by allowing W to increase a little and examining the effect on t^*, Differentiating (26) with respect to W we get

$$-[1-\pi]\,\kappa u\text{"}(W-t^*\kappa)\left[1-\kappa\frac{\partial t^*}{\partial W}\right]+[L-\kappa]\,\pi u\text{"}(W-t^*\kappa-[1-t^*]\,L)\left[1-[\kappa-L]\,\frac{\partial t^*}{\partial W}\right]=0$$

which gives

$$\frac{\partial t^*}{\partial W} = \frac{-[1-\pi]\,\kappa u\text{"}(W-t^*\kappa) + [L-\kappa]\,\pi u\text{"}(W-t^*\kappa-[1-t^*]\,L)}{[1-\pi]\,\kappa^2 u\text{"}(W-t^*\kappa) + \pi\,[L-\kappa]^2\,u\text{"}(W-t^*\kappa-[1-t^*]\,L)}$$

The denominator of this must be negative if Anne and Beth are risk averse: $u\text{"}(\cdot)$ is everywhere negative and the other terms are positive. The numerator is positive if DARA holds: therefore higher wealth reduces the demand for insurance coverage.

Solutions LT

■ 1. Payoffs

Consider the following stage game:

$1\backslash 2$	L	R
U	$2,3$	$0,1$
D	$1,0$	$5,4$

(a) (8 points) What is the set of *feasible and individually rational payoffs* associated with the infinitely repeated game with the above stage game

Solution: (a) Minmax payoffs are $\{10/6, 2\}$. Hence, it is the polygon with vertices $\{10/6, 2\}$, $\{10/6, 16/6\}$, $\{2, 3\}$, $\{5, 4\}$, and $\{3, 2\}$.

■ 2. Adverse Selection

A friend wants to buy a special pen for her father's birthday, the "16 K." Even though the list price in the collector's manual is only £250, she is willing to pay £500 for it.

She finds the pen on a new website in which people post descriptions of the items that they would like to buy and how much they are willing to pay for them. However, she is aware that some sellers might try and fool her by sending the "12 K" instead of the "16 K" and she values the 12 K at only £100. Genuine sellers would meet her offer so long as it was at least as high as the list price in the collector's manual, whereas fraudulent sellers would send her the "12 K" version if she posted any price greater than £100.

From speaking to friends, she estimates that out of all potential sellers 25% are genuine sellers and the remainder fraudulent. She determines therefore to post a price of £200(=0.75*100+0.25*500).

(a) (2 points) She told you about this, and you suggest that she is better off not posting to this site. Why?

(b) (3 points) Suppose that the proportion of potential sellers who are genuine is x. What is the smallest value of x at which Fran should post an offer at the site? At what price?

(c) (3 points) Suppose that the proportion of potential sellers who are genuine is 75%. At what price should she post an offer to the site?

Solution: (a) Since the price of £200 is less than the list price in the collector's manual, only fraudulent sellers will accept her offer. Hence, she would have a negative payoff.

(b) We want $100(1-x)+500x \geq 250$. Hence $x^* = 3/8$ is the smallest value.

(c) She should post at £250.

■ 3. Nash Equilibrium

(a) (8 points) Construct a simultaneous-move game with two players and two actions for each player that has two Nash equilibria in pure strategies and no Nash equilibrium in mixed strategies.

Solution: Example

$1\backslash 2$	L	R
U	$1,1$	$0,0$
D	$0,0$	$0,0$

Key feature: one Nash equilibrium is in weakly dominated strategies.

■ 4. Games with Sequential Moves

Suppose that the inverse market demand is given by $P = 100 - (q_i + q_e)$, where P is the market price, q_i is the output of the incumbent firm and q_e is the output of a potential entrant to the market. The incumbent firm's total cost function is $C(q_i) = 40q_i$, whereas the cost function of the entrant is $C(q_e) = K + 40q_e$, where K is a sunk cost incurred to enter the market. The incumbent chooses q_i units of output, the entrant observes it and expects this output level to be maintained.

(a) (4 points) Represent the competition described above as an extensive form game. What are the strategies of the two firms?

(b) (4 points) Compute the best response for the entrant.

(c) (4 points) Suppose that $K = 100$. How much output would the incumbent firm have to produce to just keep the entrant out of the market? What is the price at that output level?

(d) (8 points) Find the minimal K^* such that for any $K \leq K^*$ the incumbent prefers the entrant to enter the market.

Solution: (a) The set of players is $N = \{1,2\}$; the strategy choice for the incumbent is $q_i \geq 0$, the strategy of the entrant is quantity $q_e(q_i)$ as a function of the incumbent choice; Payoff of the incumbent is $\pi_e = (p-40)q_e$ and of the entrant is $\pi_i = (p-40)q_i - K$.

(b) The entrant chooses its output q_e to solve $max((100-q_i-q_e)q_e - 40q_e)$. The first order condition is equal to $100 - q_i - 40 - 2q_e = 0$, and the solution is $q_e = 30 - \frac{q_i}{2}$.

(c) The entrant's profit are

$$(100 - q_i - 30 + \frac{q_i}{2})(30 - \frac{q_i}{2}) - 40(30 - \frac{q_i}{2}) - 100 = (30 - \frac{q_i}{2})^2 - 100.$$

Thus, by choosing q_i such that $(30 - \frac{q_i}{2})^2 - 100 = 0$, the incumbent can keep the entrant out. Thus, $q_i = 40$. Since the entrant is not entering, $P = 60$.

(d) The entrant's profit are

$$(100 - q_i - 30 + \frac{q_i}{2})(30 - \frac{q_i}{2}) - 40(30 - \frac{q_i}{2}) - K = (30 - \frac{q_i}{2})^2 - K.$$

Thus, by choosing q_i such that $(30 - \frac{q_i}{2})^2 - K = 0$, the incumbent can keep the entrant out. Thus, $q_i = 60 - 2\sqrt{K}$, $P = 100 - 60 + 2\sqrt{K}$. The incumbent profits are $(100 - 60 + 2\sqrt{K} - 40)(60 - 2\sqrt{K}) = 120\sqrt{K} - 4K$.

If the incumbent accommodates entry, the incumbent maximizes:

$$(100 - q_i - 30 + \frac{q_i}{2} - 40)q_i.$$

The first order condition implies $q_i = 30$. Thus, profits equal 450.

Hence, accommodating yields higher profits than deterring entry if $450 \geq 120\sqrt{K} - 4K$. Therefore for $K \leq K^*$ such that $\sqrt{K^*} \leq \frac{120 - \sqrt{120^2 - 16 \cdot 450}}{8}$, the incumbent accommodates entry.

■ 5. Repeated Games

Consider the following normal form game:

1\2	L	C	R
T	1, 1	0, 0	1, 0
M	0, 7	5, 5	2, 1
B	0, 0	7, 0	3, 3

(a) (4 points) Identify the *set of pure strategy Nash equilibria* of this game.

(b) (4 points) Identify the *mixed strategy Nash equilibria* of this game.

(c) (6 points) Assume now that this game is played in three consecutive periods. The two players have the same discount factor δ. The average discounted payoff of the players is:

$$\Pi_i = \frac{1}{1+\delta+\delta^2}\left[g_i(a_i^1, a_{-i}^1) + \delta g_i(a_i^2, a_{-i}^2) + \delta^2 g_i(a_i^3, a_{-i}^3)\right] \tag{1}$$

where $g_i(a_i^t, a_{-i}^t)$ is the stage game payoff of player i if the strategy profile chosen by both players in period $t \in \{1, 2, 3\}$ is (a_i^t, a_{-i}^t): $a_1^t \in \{T, M, B\}$ and $a_2^t \in \{L, C, R\}$.
Construct strategies for the three-period repeated game that support the payoff $(3, 3)$ in each period of the game as a *Subgame Perfect equilibrium*. For what values of the discount factor δ are these strategies subgame perfect?

(d) (6 points) Construct strategies for the three-period repeated game that support the payoff $(5, 5)$ in periods $t = 1$ and $t = 2$, and the payoff $(3, 3)$ in period $t = 3$ for both players as a *Subgame Perfect equilibrium*. For what values of the discount factor δ are these strategies subgame perfect?

Solution: (a) The game has two pure strategy Nash equilibria: (T, L) and (B, R).

(b) In addition to the two pure strategy (degenerate mixed strategy) Nash equilibria derived in (a) above the game has the mixed strategy Nash equilibrium

$$\left(\sigma_1(U) = \frac{3}{4}, \sigma_1(M) = 0; \sigma_2(L) = \frac{2}{3}, \sigma_2(C) = 0\right)$$

with associated expected payoffs $\Pi_1(\sigma) = 1$ and $\Pi_2(\sigma) = 3/4$.

(c) The history independent strategies that support the payoff vector $(3, 3)$ are:

for player 1: play B every period independently of the history,

for player 2: play R every period independently of the history.

These strategies are a Subgame Perfect equilibrium of the three-period repeated game for all values of the discount factor δ since each player is choosing the stage game Nash equilibrium strategy in every subgame.

(d) The history dependent strategies that support the payoff vectors $(5, 5)$ in $t = 1$ and $t = 2$, and $(3, 3)$ in $t = 3$ are:

for player 1: play M in in $t = 1$.

In $t = 2$ play M if the previous period outcome is (M, C), otherwise play T.

In $t = 3$ play B if the history of outcomes is $\{(M, C), (M, C)\}$, otherwise play T.

for player 2: play C in in $t = 1$.

In $t = 2$ play C if the previous period outcome is (M, C), otherwise play L.

In $t = 3$ play R if the history of outcomes is $\{(M, C), (M, C)\}$, otherwise play L.

These strategies are a Subgame Perfect equilibrium of the three-period repeated game since each player is choosing the stage game Nash equilibrium strategy in every subgame at $t = 3$. Moreover, both players do not want to deviate from these strategies at $t = 2$ if

$$5 + \delta 3 \geq 7 + \delta 1$$

or

$$\delta \geq 1.$$

A fortiori, both players do not want to deviate at $t = 1$,

$$5 + \delta 5 + \delta^2 3 \geq 7 + \delta 1 + \delta^2 1.$$

■ 6. Signalling

A worker's type is $t \in \{0, 1\}$. The probability that any worker is of type $t = 1$ equals $2/3$, while the probability that $t = 0$ equals $1/3$. The productivity of a worker in a job is $(t + 1)^2$. Each worker chooses a level of education $e \geq 0$. The total cost of obtaining education level e is $C(e|t) = e^2(2 - t)$. The worker's wage is equal to his expected productivity.

(a) (7 points) Characterize all pooling perfect Bayesian equilibrium in which both types of workers choose a strictly positive education level.

(b) (7 points) Find all separating perfect Bayesian equilibria.

(c) (6 points) Which separating equilibrium survives the intuitive criterion? Is it the one with the lowest education level?

Solution: (a) In a pooling PBE, the two types of workers choose the same education level, denote it by $\overline{e}$. It implies that the equilibrium posterior beliefs when $e = \overline{e}$ is observed are the same as the prior beliefs. Hence the wage paid when $e = \overline{e}$ is observed equals the expected (prior) productivity, namely:

$$(2/3)4 + (1/3)1 = 3$$

Assume that in the proposed pooling PBE that if an education level e is observed which is not equal to $\overline{e}$, then it is conjectured that the worker's type is low ability and hence wage equal to 1 is offered. Given this, the incentive compatibility conditions that ensure that neither type has an incentive to deviate are as follows. For the high productivity type, it must be that for any education level e:

$$3 - \overline{e}^2 \geq 1 - e^2 \quad \Leftrightarrow \quad 3 - \overline{e}^2 \geq 1$$

since the RHS is maximized at $e = 0$. Hence, high-type's IC is satisfied if and only if $\overline{e} \leq \sqrt{2}$.

For the low type, it must be that for any education level e:

$$3 - 2\overline{e}^2 \geq 1 - 2e^2 \quad \Leftrightarrow \quad 3 - 2\overline{e}^2 \geq 1$$

since the RHS is maximized at $e = 0$. Hence, low-type's IC is satisfied if and only if $\overline{e} \leq 1$.

In summary, both types' IC conditions are satisfied provided that $\overline{e} \leq 1$. Hence, there exists a continuum of pooling PBE in which both types choose the same, strictly positive education e; for any level e less than or equal to 1.

(b) In a separating PBE, the two types of workers choose different education levels, e_H and e_L, where the former is the education level of the high type, with $t = 1$ and the former of the low type, $t = 0$. It also implies that the equilibrium posterior beliefs when $e = e_H$ is observed is that the worker is of high type with probability one, and when $e = e_L$ is observed the probability that the worker is low ability is one. Hence, the wage paid when $e = e_H$ is observed equals 4 and when $e = e_L$ is observed is 1. Assume that in the proposed separating PBE that if an education level e is observed which is neither e_H nor e_L, then it is conjectured that the worker's type is low ability and hence wage equal to 1 is offered. Given this, the incentive compatibility conditions that ensure that neither type has an incentive to deviate are as follows. For the high type, it must be that for any $e \neq e_H$:

$$4 - e_H^2 \geq 1 - e^2 \quad \Leftrightarrow \quad 4 - e_H^2 \geq 1$$

since the RHS is maximized at $e = 0$. Hence, high-type's IC is satisfied if and only if $e_H \leq \sqrt{3}$.

For the low type, it must be that for any $e \neq e_H$:

$$1 - 2e_L^2 \geq 1 - 2e^2 \quad \Leftrightarrow \quad 1 - 2e_L^2 \geq 1$$

since the RHS is maximized at $e = 0$. Hence, low-type's IC is satisfied if and only if $e_L = 0$. Moreover we need to check that the low type has no incentive to mimic the high type. That is:

$$1 - 2e_L^2 \geq 4 - 2e_H^2 \quad \Leftrightarrow \quad 2e_H^2 \geq 3 + 2e_L^2$$

Since $e_L = 0$, the latter inequality simply requires that the high-type's IC is satisfied if and only if $e_H \geq \sqrt{3/2}$.

Therefore all the separating PBE are characterized by education levels $e_H \in \left[\sqrt{3/2}, \sqrt{3}\right]$ and $e_L = 0$.

(c) The only separating equilibrium that survives the intuitive criterion is the one with the lowest education level $e_H = \sqrt{3/2}$. This is the case since the rational beliefs for the firms require that:

$$\Pr(t = 1|e) = \begin{cases} 1 & \text{if} \quad e > \sqrt{3/2} \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

as only high type workers would choose an education level $e > \sqrt{3/2}$, since:

$$\begin{aligned} 4 - e^2 &\geq 1 \\ 4 - 2e^2 &\leq 1 \end{aligned}$$

But if such are the beliefs of the firms, the high type never has an incentive to acquire more than $e_H = \sqrt{3/2}$ years of education, as he would be recognized and would receive the high wage in any event.