# Solutions EC202 Exam

## Part A

1. *Outline Answer*

   (a) The problem can be written as

$$\min w_1 z_1 + w_2 z_2 \text{ subject to } \; 4\log q \le \log(z_1 - 1) + \log(z_2 - 1).$$

with associated Lagrangian

$$w_1 z_1 + w_2 z_2 + \lambda\left[4\log q - \log(z_1 - 1) - \log(z_2)\right].$$

The first-order conditions for an interior optimum are

$$\begin{aligned} w_1 - \frac{\lambda^*}{z_1^* - 1} &= 0, \\ w_2 - \frac{\lambda^*}{z_2^* - 1} &= 0, \\ \log(z_1^* - 1) + \log(z_2^* - 1) &= 4\log q. \end{aligned}$$

Substituting from the first two FOC into the third we get

$$\begin{aligned} \log\left(\frac{\lambda^*}{w_1}\right) + \log\left(\frac{\lambda^*}{w_2}\right) &= 4\log q, \\ 4\log q + \log(w_1 w_2) &= 2\log\lambda^*, \end{aligned}$$

so that

$$\lambda^* = q^2\sqrt{w_1 w_2}.$$

Using the FOC in this expression we get obtain the the cost function $C(\mathbf{w}, q)$ as

$$\begin{aligned} w_1 z_1^* + w_2 z_2^* &= w_1 + w_2 + 2\lambda^* \\ &= w_1 + w_2 + 2q^2\sqrt{w_1 w_2}. \end{aligned}$$

   (b) From part (a) we get average and marginal cost respectively as

$$\frac{C(\mathbf{w}, q)}{q} = \frac{w_1}{q} + \frac{w_2}{q} + 2q\sqrt{w_1 w_2},$$

$$\frac{\partial C(\mathbf{w}, q)}{\partial q} = 4q\sqrt{w_1 w_2}.$$

So MC cuts AC where

$$\frac{w_1}{q} + \frac{w_2}{q} + 2q\sqrt{w_1 w_2} = 4q\sqrt{w_1 w_2},$$

which occurs at a value $\underline{q}$ such that

$$\underline{q} = \left[\frac{w_1 + w_2}{w_2}\right]^{\frac{1}{4}} / \sqrt{2}.$$

We know that AC is at a minimum at $\underline{q}$. The supply curve for the competitive firm is determined by two conditions (i) $p =$MC if output is positive and (ii) $p$ must cover AC. The competitive firm will supply a positive amount of output if $q \geq \underline{q}$. Equating price to MC we have $p = 4q\sqrt{w_1 w_2}$. So the supply curve is

$$q = \begin{cases} kp & \text{if } p > \underline{p}, \\ 0 \text{ or } \underline{q} & \text{if } p = \underline{p}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\underline{p} := 4\underline{q}\sqrt{w_1 w_2}$ and $k := \underline{q}/\underline{p}$.

2. *Outline Answer*

(a) The indifference curve is

$$-\frac{1}{y_0} - \frac{1}{y_1} = c$$

Differentiating and rearranging to get the MRS:

$$\text{MRS} = -\frac{\mathrm{d}y_1}{\mathrm{d}y_0} = \frac{y_1^2}{y_0^2} > 0.$$

Therefore

$$\frac{\mathrm{d}\,\text{MRS}}{\mathrm{d}y_0} = -2\frac{y_1^2}{y_0^3} < 0.$$

So indifference curves have "conventional" shape – convex towards the origin which implies risk aversion.

(b) The preferences exhibit constant relative risk aversion, which means that absolute risk aversion must be decreasing. we can also see this by noting that the felicity function is $u(y) = -1/y$ with first and second derivatives given respectively by $u_c(y) = 1/y^2$ and $u_{cc}(y) = -2/y^3$ . Absolute risk aversion is given by

$$-\frac{u_{cc}(y)}{u_c(y)} = \frac{2y^2}{y^3} = \frac{2}{y},$$

which is clearly decreasing in $y$.

(c) The value of the uncertain prospect is $-500^{-1} - 1\,500^{-1}$ and the value of an amount $\xi$ received with certainty is $-\xi^{-1} - \xi^{-1}$. So the certainty equivalent is the value of $\xi$ satisfying

$$2\xi^{-1} = 500^{-1} + 1\,500^{-1}$$

which means the certainty equivalent is \$750. Given that there is equal weight attached to the utility in each of the two states the implied probabilities are $\left(\frac{1}{2}, \frac{1}{2}\right)$ so that the expected payoff is $\frac{1}{2} \times \$500 + \frac{1}{2} \times \$1\,500 = \$1\,000$. Therefore the risk premium is $\$1\,000 - \$750 = \$250$.

3. *Outline Answer*

   (a) False. If there are consumption or production externalities then a CE may be inefficient.

   (b) False. If incomes are equal then the competitive equilibrium allocation is fair in the sense that no agent envies the consumption of any other agent.

   (c) False. For changes in social welfare to be proportional to changes in national income we need to be at a global optimum where the social marginal value of a dollar of income is equated across all individuals.

4. *Outline Answer*

   (a) Both player 1 and 2 have a strictly dominant strategy. In particular player 1's dominant strategy is $M$ while player 2's dominant strategy is $L$.

   (b) The unique *Dominant Strategy equilibrium* of the normal form game is $(M, L)$ with associated payoff $(5, 4)$.

5. *Outline Answer*

   (a) Player 1's best reply is:

$$BR_1(L) = BR_1(C) = BR_1(R) = \{U, M, D\}$$

   (b) Player 2's best reply is:

$$BR_2(U) = BR_2(M) = BR_2(D) = \{L\}$$

   (c) There exists three pure strategy Nash equilibria of this normal form game:

$$(U, L) \quad (M, L) \quad (D, L)$$

6. *Outline Answer*

(a) True. All non-degenerate mixed strategy Nash equilibria are such that each player is indifferent among all the pure strategies he plays with positive probability. By definition of strict dominance, a player cannot be indifferent between playing a strictly dominated strategy and an undominated strategy.

(b) False. Nash equilibria of an extensive form full information game can be supported by non-credible threats. Subgame perfect equilibria are the ones that rule out the use of non-credible threats off the equilibrium path.

(c) False. The Pooling equilibrium of a signaling game is such that the strategy choice of the informed player does not signal his private information to the uninformed player. The uninformed player beliefs then coincide with the priors.

# Part B

7. *Outline Answer*

   (a) Average costs are $\alpha_0/q+\alpha_1+\alpha_2 q$. and marginal costs are $\alpha_1+2\alpha_2 q$. Average revenue is simply $\beta_1-\beta_2 q$; therefore total revenue at $q$ is $\beta_1 q-\beta q^2$ and so, differentiating this with respect to $q$, marginal revenue is $\beta_1-2\beta_2 q$.

   (b) The first-order condition for the monopolist is given by MC=MR so that

   $$\alpha_1+2\alpha_2 q=\beta_1-2\beta_2 q$$

   from which the solution $q^*$ follows.

   (c) Substituting for $q^*$ we also get

   $$c^*=\beta_1-2\beta_2 q^*=\frac{\alpha_2\beta_1+\alpha_1\beta_2}{\beta_2+\alpha_2}$$

   $$p^*=\beta_1-\beta_2 q^*=c^*+\frac{1}{2}\beta_2\frac{\beta_1-\alpha_1}{\alpha_2+\beta_2}$$

   Given the assumption $\beta_1>\alpha_1$ we have $p^*>c^*$.

   (d) The unregulated price of the monopolist would exceed $\bar{p}$ if $q$ were below $\bar{q}$, where $\bar{q}:=[\beta_1-\bar{p}]/\beta_2$: Clearly we now have

   $$\text{AR}(q)=\left\{\begin{array}{l}\bar{p}\text{ if }q\leq\bar{q}\\ \beta_1-\beta_2 q\text{ if }q\geq\bar{q}\end{array}\right\}$$

   average revenue is a continuous function of $q$ but has a kink at $\bar{q}$. If $q<\bar{q}$ marginal revenue is simply the imposed price ceiling $\bar{p}$. If $q>\bar{q}$ then the price the market would bear is below $\bar{p}$ and so that MR is again given by the answer to part (b) in this region. Therefore

   $$\text{MR}(q)=\left\{\begin{array}{l}\bar{p}\text{ if }q<\bar{q}\\ \bar{p}\text{ or }\beta_1-2\beta_2\bar{q}\text{ if }q=\bar{q}\\ \beta_1-2\beta_2 q\text{ if }q>\bar{q}\end{array}\right\}$$

   – notice that there is a discontinuity exactly at $\bar{q}$.

   (e) Let $\tilde{q}$ denote the monopolist's chosen output under regulation. If $\bar{p}=c^*$ then the MR=MC condition yields the same value of $q$as in the unregulated case, $\tilde{q}=q^*$ (but the price is forced lower). If $\bar{p}$ is increased a little then clearly the MR schedule is moved "up" relative to the MC schedule and we must have $\tilde{q}>q^*$. However if $\bar{p}$ is increased to $p^*$ (or above) the price ceiling is irrelevant: the situation facing the monopolist is effectively the same as in the unregulated case: so $\tilde{q}=q^*$ (and the the price remains unchanged too).

8. *Outline Answer*

(a) The right thing for the consumer to do is to maximise net income. By definition net income is

$$M = y_1 - z + \frac{y_2 + \tau\left[1 - e^{-z}\right]}{1 + r}$$

The FOC for an interior maximum is

$$\frac{\partial M}{\partial z} = \frac{\tau e^{-z}}{1+r} - 1 = 0$$

which would imply

$$z = \log\left(\frac{\tau}{1+r}\right).$$

But the interior solution is only valid if $\tau > 1 + r$. So the solution is .

$$z^*(r, \tau) := \begin{cases} \log\left(\frac{\tau}{1+r}\right). & \text{if } \tau > 1 + r. \\ 0 & \text{otherwise} \end{cases}.$$

(b) If if $\tau > 1 + r$. holds then $z^*$ is decreasing in $r$ and increasing in $\tau$ An increase in individual talent $\tau$ increases the marginal return of investment and hence promotes more education; the opposite effect is caused by an increase in $r$, which makes the individual richer in the future without much investment therefore $z$ declines).

(c) Assuming $\tau > 1 + r$ , maximised income is:

$$\begin{aligned} M &= M^*(r, \tau) := y_1 - \log\left(\frac{\tau}{1+r}\right) + \frac{y_2 + \tau\left[1 - e^{-z}\right]}{1+r} \\ &= y_1 + \log\left(\frac{1+r}{\tau}\right) - 1 + \frac{y_2 + \tau}{1+r} \end{aligned}$$

Hence optimal period-1 consumption is

$$x_1^* = \alpha M = \alpha\left[y_1 + \log\left(\frac{1+r}{\tau}\right) - 1 + \frac{y_2 + \tau}{1+r}\right].$$

Borrowing is given by

$$B^*(r, \tau) := x_1^* + z^*(r, \tau) - y_1$$

Using the formulas for $z^*(r, \tau)$ and $x_1^*$ we have

$$B^*(r, \tau) = \alpha\left[\frac{y_2 + \tau}{1+r} - 1\right] + [1 - \alpha]\left[\log\left(\frac{\tau}{1+r}\right) - y_1\right].$$

(d) Differentiating we have

$$\begin{aligned} \frac{\partial B^*(r, \tau)}{\partial r} &= -\alpha\frac{y_2 + \tau}{[1+r]^2} - \frac{1-\alpha}{1+r} < 0, \\ \frac{\partial B^*(r, \tau)}{\partial \tau} &= \frac{\alpha}{1+r} + \frac{1-\alpha}{\tau} > 0. \end{aligned}$$

9. *Outline Answer*

(a) For consumers of type $a$ the Lagrangian is

$$-\frac{1}{2}\left[x_1^a\right]^{-2}-\frac{1}{2}\left[x_2^a\right]^{-2}+\lambda\left[10p-px_1^a-x_2^a\right]$$

where $p$ is the price of good 1 in terms of good 2. The FOC for a maximum are

$$\begin{aligned}\left[x_1^a\right]^{-3}-p\lambda &= 0\\ \left[x_2^a\right]^{-3}-\lambda &= 0\end{aligned}$$

Rearranging and using the budget constraint we get

$$\begin{aligned}x_1^a &= p^{-1/3}\lambda^{-1/3}\\ x_2^a &= \lambda^{-1/3}\end{aligned}$$

$$px_1^a+x_2^a=\left[p^{2/3}+1\right]\lambda^{-1/3}=10p$$

So

$$x_2^a=\lambda^{-1/3}=\frac{10p}{p^{2/3}+1}$$

For consumers of type $b$ the Lagrangian is

$$\log x_1^b+\log x_2^b+\mu\left[32-px_1^b-x_2^b\right]$$

The FOC for a maximum are

$$\begin{aligned}\left[x_1^b\right]^{-1}-p\mu &= 0\\ \left[x_2^b\right]^{-1}-\mu &= 0\end{aligned}$$

Rearranging and using the budget constraint we get

$$\begin{aligned}x_1^b &= \frac{1}{p\mu}\\ x_2^b &= \frac{1}{\mu}\end{aligned}$$

$$px_1^b+x_2^b=\frac{2}{\mu}=32$$

So

$$x_2^b=\frac{1}{\mu}=16$$

.The excess demand function for good 2 is therefore

$$E_2(p)=\frac{10np}{p^{2/3}+1}-16n.$$

Similarly, the excess demand function for good 1 is

$$E_1(p)=\frac{16n}{p}-\frac{10n}{p^{2/3}+1}.$$

(b) From part (a) we can see that excess demand for each good is zero where $\frac{5}{8}p - p^{2/3} - 1 = 0$. Clearly $p = 8$satisfies this condition and so constitutes a competitive equilibrium

(c) Differentiating the excess demand function for good 2.

$$\frac{dE_2(p)}{dp} = 10\frac{\left[p^{2/3}+1\right] - p\left[\frac{2}{3}p^{-1/3}\right]}{\left[p^{2/3}+1\right]^2} = 10\frac{\frac{2}{3}p^{2/3}+1}{\left[p^{2/3}+1\right]^2} > 0$$

for all values of $p$. So $E_2$ is always increasing in the price of good 1 (decreasing in the price of good 2). Therefore the excess demand for good 2 is always positive if $p > 8$ and is always negative if $p < 8$. The system is globally stable under tï¿œtonnement, The same reasoning implies that there can be no other value of $p$ such that excess demand is zero. So the equilibrium is unique.

(d) Using the demand functions in part (a) we can see that at the competitive equilibrium with $p = 8$ each $a$ type consumes $(8, 16)$ and each $b$ type consumes $(2, 16)$. If $n$ is large then this is the only allocation in the core.

# Part C

10. *Outline Answer*

(a) The normal form game is:

| $1\backslash 2$ | $L$ | $R$ |
|---|---|---|
| $U$ | $5,5$ | $3,9$ |
| $M$ | $9,5$ | $5,5$ |
| $D$ | $3,6$ | $0,1$ |

Clearly strategies $U$ and $D$ for player 1 are strictly dominated. In other words player 1 has a strictly dominant strategy $M$.

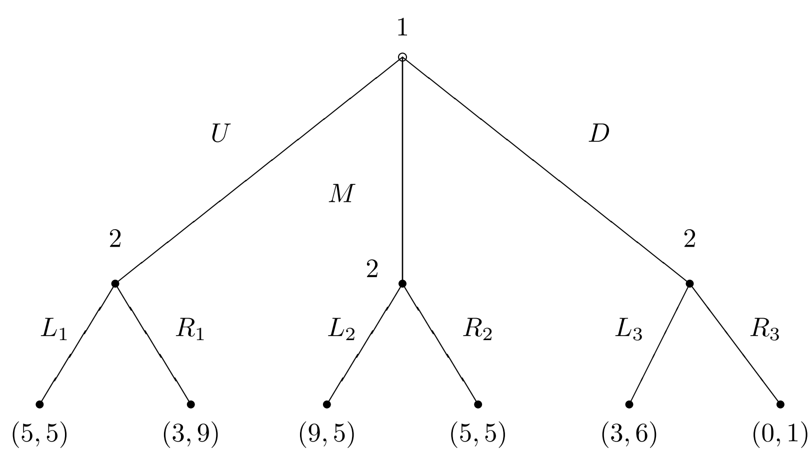
(b) There exists two pure strategy Nash equilibria of this game: $(M, L)$ and $(M, R)$ with payoffs $(9, 5)$ and $(5, 5)$ respectively.

(c) There exists a continuum of additional mixed strategy Nash equilibria of this game:

$$(\sigma_1(M) = 1; \sigma_2(L) = q \in [0, 1])$$

with expected payoffs $(5 + 4q, 5)$.

(d) In the extensive form game the strategy of player 1 is $s_1 \in \{U, M, D\}$ while the strategy of player 2 is the vector $s_2 = (s_2(U), s_2(M), s_2(D))$ where $s_2(U) \in \{L_1, R_1\}$, $s_2(M) \in \{L_2, R_2\}$ and $s_2(D) \in \{L_3, R_3\}$. The extensive form game can be represented as:




The normal form associated with the extensive form of this dynamic game is:

| $1\backslash 2$ | $L_1, L_2, L_3$ | $L_1, L_2, R_3$ | $L_1, R_2, L_3$ | $L_1, R_2, R_3$ | $R_1, L_2, L_3$ | $R_1, R_2, L_3$ | $R_1, L_2, R_3$ | $R_1, R_2, R_3$ |
|---|---|---|---|---|---|---|---|---|
| $U$ | $5,5$ | $5,5$ | $5,5$ | $5,5$ | $3,9$ | $3,9$ | $3,9$ | $3,9$ |
| $M$ | $9,5$ | $9,5$ | $5,5$ | $5,5$ | $9,5$ | $5,5$ | $9,5$ | $5,5$ |
| $D$ | $3,6$ | $0,1$ | $3,6$ | $0,1$ | $3,6$ | $3,6$ | $0,1$ | $0,1$ |

1. *quad*

   (a) (e) There exist eight pure strategy Nash equilibria of this extensive form game. These are all characterized by player 1's strategy choice $s_1^* = M$ and by player 2's strategy choices $s_2^* = (\alpha, \beta, \gamma)$ with $\alpha \in \{L_1, R_1\}$, $\beta \in \{L_2, R_2\}$ and $\gamma \in \{L_3, R_3\}$.

   (b) (f) There exist two (pure strategy) subgame perfect equilibria of the extensive form game: $[M, (R_1, L_2, L_3)]$ with associated payoff $(9, 5)$ and $[M, (R_1, R_2, L_3)]$ with associated payoff $(5, 5)$

11. *Outline Answer*

(a) The normal form game is such that:

- player 1's strategy $M$ is strictly dominated (by his strategy $D$),
- player 2's strategy $C$ is strictly dominated (by his strategy $R$).

(b) The game has two pure strategy Nash equilibria $NE^1 = (U, L)$ and $NE^2 = (D, R)$ both with payoff $(4, 4)$.

(c) Notice first that since strategy $M$ is strictly dominated for player 1 and strategy $C$ is strictly dominated for player 2 there does not exists a non-degenerate mixed strategy Nash equilibrium where player 1 allocated positive probability on strategy $M$ and player 2 allocated strictly positive probability on strategy $C$. In other words all mixed strategy equilibria are such that $\sigma_1(M) = 0$ and $\sigma_2(C) = 0$.

The unique (non-degenerate) mixed strategy Nash equilibrium if this game is then such that:

$$NE^3 = \left(\sigma_1(U) = \frac{2}{3}, \sigma_1(M) = 0; \sigma_2(L) = \frac{2}{3}, \sigma_2(C) = 0\right)$$

with associated payoff

$$E\Pi^3 = \left(\frac{8}{3}, \frac{8}{3}\right)$$

(d) The history independent strategies that support the payoff vector $(4, 4)$ are:

- quad *for player 1:* play $U$ every period independently of the history,
- quad *for player 2:* play $L$ every period independently of the history.

Of course the following strategies reach exactly the same outcome:

- quad *for player 1:* play $D$ every period independently of the history,
- quad *for player 2:* play $R$ every period independently of the history.

These strategies are a Subgame Perfect equilibrium of the two-periods repeated game for all values of the discount factor $\delta$ since each player is choosing the stage game Nash equilibrium strategy in every subgame.

(e) The history dependent strategies that support the payoff vectors $(5, 5)$ in $t = 1$ and $(4, 4)$ in $t = 2$ are:

- *for player 1:* play $M$ in in $t = 1$.

  In $t = 2$ play $U$ if the previous period outcome is $(M, C)$, otherwise play the mixed strategy $(\sigma_1(U) = \frac{2}{3}, \sigma_1(M) = 0)$.

- *for player 2:* play $C$ in in $t = 1$.

  In $t = 2$ play $L$ if the previous period outcome is $(M, C)$, otherwise playthe mixed strategy $(\sigma_2(L) = \frac{2}{3}, \sigma_2(C) = 0)$.

Clearly the strategies above can be written substituting $U$ with $D$ and $L$ with $R$.

These strategies are a Subgame Perfect equilibrium of the two-periods repeated game since each player is choosing the stage game Nash equilibrium strategy in every subgame at $t = 2$.

Moreover, both players do not want to deviate from these strategies if

$$5 + \delta 4 \geq 6 + \delta \frac{8}{3}$$

or

$$\delta \geq \frac{3}{4}$$

12. *Outline Answer*

(a) Cournot competition can be represented as the following game. The set of players is $N = \{1, 2\}$, the strategy choice for player $i \in \{1, 2\}$ is $\{q_i \geq 0\}$, payoff for player $i \in \{1, 2\}$ is $\Pi_i = q_i(p - c)$.

(b) Best reply for firm $i \in \{1, 2\}$ solves:

$$\max_{q_i} \quad q_i(1 - q_i - q_{-i})$$

and is:

$$q_i = \frac{(1 - q_{-i})}{2}$$

The unique Nash equilibrium of this game is characterized by strategies:

$$q_1^c = q_2^c = \frac{1}{3}$$

and profits:

$$\Pi_1^c = \Pi_2^c = \frac{1}{9}$$

(c) The monopolist problem is:

$$\max_{Q} \quad Q(1 - Q)$$

and is:

$$Q^m = \frac{1}{2}, \qquad \Pi^m = \frac{1}{4}$$

(d) The reason

$$q_1^m = q_2^m = \frac{1}{4}$$

is not a Nash equilibrium of the Cournot game is that each firm has a profitable deviation. Let $q_{-i} = 1/4$ then the quantity

$$\bar{q}_i = \frac{3}{8}$$

generate profits higher than half of the monopolist profits:

$$\pi_i\left(\bar{q}_i, q_{-i}^m\right) = \frac{9}{64} > \pi_i\left(q_i^m, q_{-i}^m\right) = \frac{1}{8}$$

(e) The history dependent strategies that support the firms choice of half of the monopolist quantity $q_i^m = 1/4$ are for firm $i$:

- Choose $q_i^m$ at $t = 1$.
- Continue to choose $q_i^m$ if the previous period outcome is $(q_i^m, q_{-i}^m)$.

- If in the previous period the outcome is such that $q_j \neq q_j^m$ for either $j \in \{1, 2\}$ than choose $q_i^c$ for the remainder of the game.

Clearly these trigger strategies are a Subgame Perfect equilibrium of any punishment sub game since they prescribe for each player to choose the Cournot quantities, the stage game Nash equilibrium strategies, in every period.

Moreover, both players do not want to deviate from these strategies if

$$\pi_i\left(q_i^m, q_{-i}^m\right) = \frac{1}{8} \geq (1-\delta)\pi_i\left(\bar{q}_i, q_{-i}^m\right) + \delta\pi_i\left(q_i^c, q_{-i}^c\right) = (1-\delta)\frac{9}{64} + \delta\frac{1}{9}$$

or

$$\delta \geq \frac{9}{17}$$