SOLUTIONS Summer 2013 Examination

EC202

Microeconomic Principles II

Section A

1. ___.

(a) (i) The Lagrangean for cost minimisation is

$$w_1 z_1 + w_2 z_2 + \lambda \left[\log q - {}^1\!/\!_2 \log(z_1 - a) - {}^1\!/\!_2 \log(z_2)\right]$$

which has FOCs

$$w_1 = \frac{{}^1\!/\!_2\lambda}{z_1 - a},\ w_2 = \frac{{}^1\!/\!_2\lambda}{z_2}$$

Using the production constraint the FOCs yield

$$\log q = {}^1\!/\!_2 \log(z_1 - a) + {}^1\!/\!_2 \log(z_2) = {}^1\!/\!_2 \log\left(\frac{{}^1\!/\!_2\lambda}{w_1}\right) + {}^1\!/\!_2 \log\left(\frac{{}^1\!/\!_2\lambda}{w_2}\right)$$

So that

$$\lambda = 2q\sqrt{w_1 w_2}$$

Minimised cost is

$$w_1 z_1 + w_2 z_2 = a w_1 + \lambda$$

So the cost function is

$$C(w_1, w_2, q) = a w_1 + 2q\sqrt{w_1 w_2}$$

(ii) It is clear that MC is $2\sqrt{w_1 w_2}$ (a constant) and AC is $aw_1/q + 2\sqrt{w_1 w_2}$ (falling with q if a is positive and constant otherwise) (iii) IRTS if $a > 0$; CRTS if $a = 0$.

(b) ___.

i. False. A consumer's cost function must be homogenous of degree 1 in all prices.
ii. False. Symmetry only applies to the substitution effect.
iii. True.

(c) (i) Using the standard definition we have $var(x) = \mathcal{E}(x^2) - (\mathcal{E}x)^2$. So we have

$$U = \mathcal{E}u(x) = 2a\mathcal{E}x - \mathcal{E}(x^2) = 2a\mathcal{E}x - (\mathcal{E}x)^2 - var(x).$$

(ii) Differentiating the felicity function we have

$$u_x(x) = 2a - 2x,$$

$$u_{xx}(x) = -2,$$

Absolute risk aversion is given by

$$-\frac{u_{xx}(x)}{u_x(x)} = \frac{1}{a-x}$$

which is clearly increasing in x. (iii) Multiplying the above by x gives relative risk aversion, which obviously must be increasing.

(d) (i) The contours are hyperbolae (case $\epsilon = 1$), 45-degree lines (case $\epsilon \to 0$), L shapes (case $\epsilon \to \infty$). (ii) geometric mean income (case $\epsilon = 1$), mean income (case $\epsilon \to 0$), max $\hat{y}$ such that $\int_0^{\hat{y}} f(y)dy = 0$ (case $\epsilon \to \infty$).

(e) For any static game define the Pure Strategy Nash equilibrium (PNE) and the Dominant Strategy equilibrium (DSE) solution concepts. Argue why any DSE is a PNE.

Answer: A strict Dominant Strategy equilibrium of a game G consists of a strategy profile $\mathbf{a}$ such that for any $\mathbf{a}'_{-i}$ and $i \in N$:

$$u_i(a_i, \mathbf{a}'_{-i}) > u_i(a'_i, \mathbf{a}'_{-i}) \text{ for any } a'_i \in A_i$$

A (pure strategy) Nash equilibrium of a game G consists of a strategy profile $\mathbf{a} = (a_i, \mathbf{a}_{-i})$ such that for any $i \in N$:

$$u_i(\mathbf{a}) \geq u_i(a'_i, \mathbf{a}_{-i}) \text{ for any } a'_i \in A_i$$

If $\mathbf{a}$ is a DSE then $a_i \in b_i(\mathbf{a}'_{-i})$ for any $\mathbf{a}'_{-i}$ and $i \in N$. This implies $\mathbf{a}$ is PNE since $a_i \in b_i(\mathbf{a}_{-i})$ for any $i \in N$.

Comments to grader: Each Definition is worth 2 marks, the argument is worth 4 marks.

(f) Four friends wish to arrange a doubles tennis match. Each one of them can choose whether to call the tennis club make arrangements, or to defer counting on others to call. At least one of them has to call the club for the match to take place. The value of playing the match is equal to 10 for a player who did not invest time in organizing the game, and is equal to 9 for a player who called. A player's utility is 0 if the match is not played.

i. Find the mixed strategy Nash equilibrium of this game.
ii. What is the probability of the match being played?
iii. What would happen if only two players had to arrange a singles the match?

Answer: Let us look for a symmetric MNE. Let p denote the probability that any one of the four players calls the club. The probability that at least one of the n friends calls the club is then determined by

$$Pr(\text{At Least 1 Calls}) = 1 - Pr(\text{Nobody Calls}) = 1 - (1-p)^n.$$

The indifference condition that ensures that players randomize requires

$$U(Call, p) = 9 = 10[1 - (1-p)^3] = U(NoCall, p).$$

Therefore, the probability that a given player call the club is

$$p = 1 - (1/10)^{1/3},$$

while the probability that at least one of the players calls the club is

$$P = 1 - (1-p)^4 = 1 - (1/10)^{4/3}.$$

If only two players had to play, the probability that a given player call the club would be

$$p_2 = 9/10,$$

while the probability that at least one of the players calls the club is

$$P_2 = 1 - (1-p)^2 = 1 - (1/10)^2.$$

If so both players would call with higher probability as $p_2 > p$ and the game would be played with higher probability as $P_2 > P$.

Comments to grader: The doubles match equilibrium is worth 4 marks, the joint probability is worth 2 marks, and the singles problem is worth 2 marks.

(g) Consider the classical Bertrand competition model. There are two firms who produce a single output a constant marginal cost equal to 10. Demand in the market satisfies $d(p) = 20 - p$. Find the monopoly price for this market. Next suppose that the firms play the game infinitely may times. Consider the strategy that requires all firms to sell at the monopoly price provided that no other firm ever deviated in the past, and to sell at marginal cost otherwise. Find the lowest value for the discount factor for which this strategy is Subgame Perfect?

Answer: If only one firm operates in the market it would choose the price so to

$$\max_p d(p)(p-c) = \max_p (20-p)(p-10)$$

Thus a profit maximizing monopolist would sell goods at a price $p = 15$ and achieve profits equal to 25. If two firms were to collude on such pricing strategy, they would want to comply with equilibrium only if their profits exceed any possible deviation gain

$$12.5 \geq (1-\delta)D + \delta 0.$$

The profit maximizing deviation is to undercut the competitor by an infinitesimal amount and to conquer the entire market $D = 25$. Thus the discount factor must exceed 1/2 for the strategy to be SPE. The requirement is also sufficient to establish that the strategy is SPE since competition is a Nash equilibrium of the stage game and therefore a SPE of the infinite repetition.

Comments to Grader: The monopoly problem is worth 3 marks, showing that the repeated game strategy is NE is worth 3 marks, arguing perfection an additional 2 marks.

(h) A challenger contests an incumbent firm. The challenger can be weak with probability 1/3 or strong with probability 2/3. The challenger knows its type, but the incumbent does not. The challenger can decide how much effort to devote to prepare for the fight. Effort is costly. The total cost of effort level e is $C_w(e) = e^2$ for a weak player, and $C_s(e) = e^2/8$. After observing the effort level of the challenger, the incumbent can choose whether to fight the entrant. The incumbent's payoff is equal: to 2 if he fights a weak challenger; to 0 if he fights a strong challenger; and to 1 otherwise. The challenger's payoff is equal: to 1 minus the cost of effort, if he is weak and the incumbent fights; 2 minus the cost of effort, if he is strong and the incumbent fights; and is equal to 3 otherwise. Find all the separating Perfect Bayesian Equilibria of this game.

Answer: In a separating PBE, the two types of challengers choose different effort levels, e_s and e_w, where the former is the effort level of the strong type, while the latter is the effort level of the weak type. The equilibrium posterior beliefs are such that: (i) when $e = e_w$ is observed, the challenger is strong with probability one; (ii) when any other effort level e is observed (including e_w), the challenger is weak with probability one. In such equilibria the effort reveals strength, thus the incumbent prefers to fight weak challengers (as $2 > 1$), but to acquiesce to strong ones (as $0 < 1$). For such beliefs, the incentive constraint of the strong type requires him to prefer effort e_s to any other $e \neq e_s$:

$$3 - e_s^2/8 \geq 2 - e^2/8 \quad \Leftrightarrow \quad 8 - e_s^2 \geq 0$$

where the implication holds since the RHS is maximized at $e = 0$. Hence, strong-type's IC is satisfied if and only if $e_s \leq 2\sqrt{2}$.

The low type must also prefer not to deviate to any other $e \neq e_s$:

$$1 - e_L^2 \geq 1 - e^2 \quad \Leftrightarrow \quad - e_w^2 \geq 0$$

where the implication holds since the RHS is maximized at $e = 0$. Hence, low-type's IC is satisfied if and only if $e_w = 0$. Moreover, we need to check that the low type has no incentive to mimic the high type. That is:

$$1 - e_w^2 \geq 3 - e_s^2 \quad \Leftrightarrow \quad e_s^2 \geq 2 + e_w^2$$

Since $e_w = 0$, the latter inequality simply requires that the high-type's IC is satisfied if and only if $e_s \geq \sqrt{2}$.

Therefore all the separating PBE are characterized by education levels $e_H \in \left[\sqrt{2}, 2\sqrt{2}\right]$ and $e_L = 0$.

Comments to grader: Give partial credit. Students need only to find all the equilibria and to argue why these are PBE.

Section B [Answer at least ONE and no more than TWO questions]

2.

(a) The parameter α captures the degree to which the goods are seen as substitutes.

(b) Profits for firm i are

$$\Pi_i = \frac{q_i^\alpha}{K} - b - cq_i \tag{1}$$

where

$$K := \sum_{j=1}^{n} q_j^\alpha$$

The first-order condition for maximising (**??**) with respect to q_i (taking all the other q_j as given) is

$$\frac{\partial \Pi_i}{\partial q_i} = \frac{\alpha q_i^{\alpha-1}}{K} - \frac{\alpha q_i^{2\alpha-1}}{K^2} - c = 0 \tag{2}$$

(c) If all firms are identical, then in equilibrium all firms must produce the same amount and so

$$K = nq_i^{*\alpha} \tag{3}$$

Substituting (**??**) in (**??**) we get

$$\frac{\alpha}{n} - \frac{\alpha}{n^2} - cq_i^* = 0 \tag{4}$$

from which the result follows immediately.

(d) To find the elasticity of demand for firm i take logs of the inverse demand curve (in the question) and differentiate with respect to q_i

$$-\frac{q_i}{p_i}\frac{\partial p_i}{\partial q_i} = 1 - \alpha + \frac{\alpha q_i^\alpha}{K} \tag{5}$$

To find the elasticity in the neighborhood of the equilibrium substitute (**??**) in (**??**) and take the reciprocal.

(e) If$\alpha = 1$ then find that firm i's profits are

$$\Pi_i^* = \frac{q_i^*}{K} - b - cq_i^* \tag{6}$$

$$= \frac{1}{n} - b - \frac{n-1}{n^2} \tag{7}$$

$$= \frac{1}{n^2} - b \tag{8}$$

Requiring that the right-hand side of (**??**) be non-negative implies $n \leq \sqrt{b}$.

3.

(a) The budget constraint is

$$c \leq w\ell + \bar{y}$$

(b) Substitute the budget constraint into the utility function; the problem is equivalent to maximising $[w\ell + \bar{y}]^\theta + [T - \ell]^\theta$. The first-order condition for an interior maximum is:

$$w\left[w\ell + \bar{y}\right]^{\theta-1} - [T - \ell]^{\theta-1} = 0$$

and so

$$\begin{aligned} w\left[\frac{w\ell + \bar{y}}{T - \ell}\right]^{\theta-1} &= 1 \\ \frac{w\ell + \bar{y}}{T - \ell} &= w^\sigma \end{aligned}$$

where $\sigma := \frac{1}{1-\theta}$. Therefore we have

$$w\ell + \bar{y} = w^\sigma \left[T - \ell\right]$$

$$\ell = \frac{Tw^\sigma - \bar{y}}{w^\sigma + w}$$

If $\bar{y}$ were large enough then ℓ in the above formula would become negative, which is logically impossible. Hence the result. The individual would choose not to work if

$$w \leq \underline{w} := \left[\frac{\bar{y}}{T}\right]^{1/\sigma}.$$

(c) (i) If $\ell > 0$ then

$$\frac{\partial \ell}{\partial \bar{y}} = -\frac{1}{w^\sigma + w} < 0$$

Also an increase in $\bar{y}$ must increase the threshold wage level $\underline{w}$ – it will make it more likely that a person does not work at all. (ii) $\ell > 0$ then

$$\frac{\partial \ell}{\partial w} = \frac{\partial}{\partial w}\left(\frac{Tw^\sigma - \bar{y}}{w^\sigma + w}\right)$$

$$= \sigma\frac{Tw^{\sigma-1}}{w^\sigma + w} - \frac{Tw^\sigma - \bar{y}}{\left[w^\sigma + w\right]^2}\left[\sigma w^{\sigma-1} + 1\right]$$

$$= \sigma w^{\sigma-1}\frac{Tw^\sigma + w}{\left[w^\sigma + w\right]^2} - \frac{Tw^\sigma - \bar{y}}{\left[w^\sigma + w\right]^2}\sigma w^{\sigma-1} - \frac{w^\sigma - \bar{y}}{\left[w^\sigma + w\right]^2}$$

$$= \sigma w^{\sigma-1}\frac{w + \bar{y}}{\left[w^\sigma + w\right]^2} - \frac{w^\sigma - \bar{y}}{\left[w^\sigma + w\right]^2}$$

$$= \frac{\sigma w^\sigma + \sigma w^{\sigma-1}\bar{y} - w^\sigma + \bar{y}}{\left[w^\sigma + w\right]^2}$$

$$= \frac{w^\sigma\left[\sigma - 1\right] + \bar{y}\left[\sigma w^{\sigma-1} + 1\right]}{\left[w^\sigma + w\right]^2}$$

Work may decrease with the wage rate if $\sigma < 1$ (equivalently $\theta < 0$) and $\bar{y}$ is small. (iii) It is clear from the labour-supply function that changes in T have the opposite effect from changes in $\bar{y}$.

(d) Introduction of the benefit is a pure income effect and so decreases ℓ; the tax will decrease the effective wage and so, may increase or decrease ℓ. Free child care will have the effect of increasing T.

4. Notice that the term x_1^a that appears in the b-type's utility function is a negative externality: the more that a-people consume of commodity 1, the lower is every b-person's utility.

   (a) If N is large we know that equilibrium will be a price taking competitive equilibrium. To derive the CE notice that the term x_1^a is virtually irrelevant to the b-people's behaviour (they cannot do anything about it). Both a-people and b-people thus have Cobb-Douglas utility functions, and we know that, if they act as price takers, their demands will be given by:

$$x_i^{*h} = \frac{1}{2}\frac{y^h}{p_i}, h = a, b; i = 1, 2.$$

   Incomes are $y^a = 300p_1$, $y^b = 200p_2$. Using this information we can see that total demand for commodity 1 is

$$N\left[x_1^{*a} + x_1^{*b}\right] = N\left[\frac{1}{2}\cdot 300 + \frac{1}{2}\cdot\frac{200}{\rho}\right]$$

   where $\rho := p_1/p_2$ (Notice that only the price ratio matters in the solution). Clearly there are $300N$ units of commodity 1 available, so the excess demand function for commodity 1 is:

$$E_1 = N\left[150 + \frac{100}{\rho} - 300\right]$$

   By Walras' Law we know that if $E_1 = 0$ then $E_2 = 0$ also. Clearly $E_1 = 0$ when $\rho = 2/3$; this is the equilibrium price ratio. Using the demand functions we find

$$\begin{bmatrix} x_1^{*a} \\ x_2^{*a} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\cdot 300 \\ \frac{1}{2}\cdot 300\rho \end{bmatrix} = \begin{bmatrix} 150 \\ 100 \end{bmatrix} \tag{9}$$

$$\begin{bmatrix} x_1^{*b} \\ x_2^{*b} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\cdot 200/\rho \\ \frac{1}{2}\cdot 200 \end{bmatrix} = \begin{bmatrix} 150 \\ 100 \end{bmatrix} \tag{10}$$

   This is the competitive equilibrium allocation.

   (b) This is the reason that the competitive equilibrium will be inefficient. To verify that this allocation is inefficient consider the following. Suppose $k < 0$: since there is a negative externality, it is likely that in the competitive equilibrium the a-people are consuming too much of commodity 1. So let us change the allocation in such a way that the a-people consume less of commodity 1 ($\Delta x_1^a < 0$) but where the a-people's utility remains unchanged: this means that their consumption of good 2 must be increased, by an amount $\Delta x_2^a = -\rho\Delta x_1^a > 0$ (remember that in equilibrium ρ equals the marginal rate of substitution). Now since there is a fixed total amount of each commodity, the b-people's consumptions must move in exactly the opposite direction; so $\Delta x_1^b = -\Delta x_1^a > 0$ and $\Delta\ x_2^b = -\Delta x_2^a < 0$. The effect on their utility can be computed thus:

$$\begin{aligned}\Delta U^b &= \frac{1}{x_1^b}\Delta x_1^b + \frac{1}{x_2^b}\Delta x_2^b + \frac{k}{x_1^a}\Delta x_1^a \\ &= \left[\frac{k}{150} + \frac{1}{100}\cdot\frac{2}{3} - \frac{1}{150}\right]\Delta x_1^a \\ &= \frac{k}{150}\Delta x_1^a > 0\end{aligned}$$

So, as we expected, it is possible to move away from the competitive equilibrium in such a way that some people's utility is increased, and no-one else's utility decreases. A similar argument follows for the positive-externality case $k > 0$: in this case change the allocation such that $\Delta x_1^a > 0$ and adjust $\Delta x_2^a$to keep a-utility constant.

(c) If $k = 0$ then there is no externality and, for $N = 1$, the set of efficient allocations can be represented by the contract curve in an Edgeworth box. Given the initial endowment we can draw in the reservation indifference curves for a and for b; the core (the segment of the contract curve bounded by these two indifference curves) is the set of possible equilibria. The solution in part (a) is one point in the core.

Section C [Answer at least ONE and no more than TWO questions]

5. Consider two players engaged in an ultimatum game to share a surplus S. Player A can offer any share $\theta \in [0,1]$ of the surplus S to Player B. If B rejects the offer, no project is undertaken and both players receive a payoff of zero. If the offer is accepted, the payoff of player A is given by $S\theta(1-\theta)$, while the payoff of player B satisfies $S(\theta - S/10)$.

   (a) Find the Subgame Perfect equilibrium of this game. [8 Marks]

   Answer: For any offer θ received by Player B, Subgame Perfection requires accepting the offer if and only if $\theta \geq S/10$, since

   $$S(\theta - S/10) \geq 0$$

   Thus, Player A can choose whether to offer a share $\theta \geq S/10$ and achieve a payoff equal to $S\theta(1-\theta)$, or to offer a share $\theta < S/10$, and receive a payoff equal to 0. Subgame Subgame Perfection requires Player A to offer $\theta = 1/2$, so long as $1/2 \geq S/10$ (i.e. $S \leq 5$), since

   $$\begin{aligned} \frac{d}{d\theta}\theta(1-\theta) &= 1 - 2\theta = 0 \text{ for } \theta = 1/2, \\ \frac{d^2}{d\theta^2}\theta(1-\theta) &= -2 < 0 \text{ for any } \theta, \end{aligned}$$

   and to offer $\theta = S/10$, so long as $1/2 < S/10$ (i.e. $S > 5$), since

   $$\frac{d}{d\theta}\theta(1-\theta) = 1 - 2\theta \leq 0 \text{ for } \theta \geq 1/2.$$

   (b) Find a Nash equilibrium of the game that is not Subgame Perfect. [6 Marks]

   Answer: Consider a strategy by Player B that entails rejecting any offer $\theta < 1$, while accepting a share of 1; and a strategy by Player A that entails offering exactly $\theta = 1$. Such strategies are trivially NE, since no player could benefit by deviating as Player B gets his best payoff, while Player A is indifferent between all of his choices.

   (c) Next suppose that, prior to the beginning of the game, Player B can choose the size of the surplus S provided that $S \in [0, 10]$. Find the Subgame Perfect equilibrium of this game. [6 Marks]

   Answer: Now player B can choose whether to set $S \leq 5$ and receive a payoff $S(1/2 - S/10)$, or to offer a share $\theta < S/10$, and receive a payoff equal to 0 (as his participation constraint binds. Subgame Perfection then requires $S^* = 5/2$ since

   $$\begin{aligned} \frac{d}{dS}S(1/2 - S/10) &= 1/2 - S/5 = 0, \\ \frac{d^2}{dS^2}S(1/2 - S/10) &= -1/10 < 0 \text{ for any } \theta \end{aligned}$$

   and because $S^*(1/2 - S^*/10) = 5/8 > 0$. A hold-up problem constraints, but does not eliminate the opportunity to produce surplus in this environment.

6. Consider the problem of a monopoly supplier of water in a council. Supplying water has a constant marginal cost of 6£ per cubic liter. There are two types of households in the economy. The preferences of both types of households are separable in water and money. Wealthy households value x cubic liters of water according to the map $u_H(x) = 18\log(x+1)$, whereas low income households value water according to $u_L(x) = 12\log(x+1)$. Only one third of the households is wealthy. The monopolist can offer contracts which specify the maximal water supply to the household, and the total price charged for this supply.

   (a) Suppose that the water supplier knows the income status of each household. Find the revenue maximizing contracts offered to high and low income households. [7 Marks]

   Answer: If the monopolist can recognize the two type of consumers, it price discriminates both types. If offers to type t a contract (P_t, x_t). In this scenario, the firm maximizes profits subject to the participation constraints of the two types of buyers,

$$\max_{P,x} 1/3[P_H - 6x_H] + 2/3[P_L - 6x_L] \text{ s.t.}$$
$$18\log(x_H+1) - P_H \geq 0 \qquad \text{(PC(H))}$$
$$12\log(x_L+1) - P_L \geq 0 \qquad \text{(PC(L))}$$

   Since PC of both types must bind the problem simplifies to

$$\max_x 1/3[18\log(x_H+1) - 6x_H] + 2/3[12\log(x_L+1) - 6x_L] \Rightarrow$$
$$x_H^* = 2 \text{ and } x_L^* = 1.$$

   The resulting equilibrium fees are found by the PCs binding at the optimum

$$P_H^* = 18\log(3) \text{ and } P_L^* = 12\log(2).$$

   Comments to Grader: 2 marks for the monopolist problem, 5 for its solution.

   (b) Next suppose that the income status of each household is unobservable. Find the revenue maximizing contracts offered by the monopolist. [10 Marks]

   Answer With incomplete information the principal has to account not only of the participation constraints, but also of the incentive constraints. The incentive constraints of this problem satisfy

$$18\log(x_H+1) - P_H \geq 18\log(x_L+1) - P_L \qquad \text{(IC(H))}$$
$$12\log(x_L+1) - P_L \geq 12\log(x_H+1) - P_H \qquad \text{(IC(L))}$$

   The problem of the firm can be written as:

$$\max_{P,x} 1/3[P_H - 6x_H] + 2/3[P_L - 6x_L] \text{ s.t.}$$
$$18\log(x_H+1) - P_H \geq \max\{18\log(x_L+1) - P_L, 0\}$$
$$12\log(x_L+1) - P_L \geq \max\{12\log(x_H+1) - P_H, 0\}.$$

   Prior to solving the problem, notice that: PC(L) holds with equality (otw firm can increase profits raising P_L),

$$12\log(x_L+1) - P_L = 0;$$

IC(H) holds with equality (otw firm can increase profits raising P_H)

$$18\log(x_H+1)-P_H=18\log(x_L+1)-P_L;$$

PC(H) is strict (by the previous two equalities and $H>L$), and IC(L) is strict (by no pooling theorem as otw $x(H)=x(L)$). The previous remarks simplify the firm's problem to:

$$\begin{gathered}\max_{P,x} 1/3[P_H-6x_H]+2/3[P_L-6x_L] \text{ s.t.}\\ 18\log(x_H+1)-P_H=18\log(x_L+1)-P_L\\ 12\log(x_L+1)-P_L=0.\end{gathered}$$

First order conditions evaluated for our functional specification then require

$$\begin{aligned}-2+\mu 18/(x_H+1) &= 0 && \text{(x(H))}\\ -4+\lambda 12/(x_L+1)-\mu 18/(x_L+1) &= 0 && \text{(x(L))}\\ 1/3-\mu &= 0 && \text{(P(H))}\\ 2/3-\lambda+\mu &= 0 && \text{(P(L))}\end{aligned}$$

Notice that $\mu=1/3$, $\lambda=1$ and thus,

$$x_H^\circ=2 \text{ and } x_L^\circ=1/2.$$

The resulting equilibrium fees are found by the constraints binding at the optimum

$$P_H^\circ=18\log(3)-6\log(3/2) \text{ and } P_L^\circ=12\log(3/2).$$

This is inefficient since the MRS of the low type is different from the MRT of the monopolist which is 6.

Comments to Grader: 2 marks for the monopolist problem, 3 for the analysis of constraints, 5 for its solution.

(c) Quantify the information rents received by each household in part (b). [3 Marks]

Answer: Low income households remain on their participation constraint so they gain nothing from trade as their utility remains equals to zero, since

$$12\log(x_L^\circ+1)-P_L^\circ=0=12\log(x_L^*+1)-P_L^*.$$

High income household benefit instead, as they are able to consume the original bundle, but at a lower price. In particular their utility increases from zero to $6\log(3/2)$ since

$$18\log(x_H^\circ+1)-P_H^\circ=6\log(3/2)>0=18\log(x_H^*+1)-P_H^*.$$

Thus, the information rent to high income consumers is equal to $6\log(3/2)$.

Comments to Grader: 2 marks for high value households, 1 for low value households.

7. Consider a Principal-Agent problem with: two exogenous states of nature $\{A, B\}$; three effort levels $\{e_h, e_m, e_l\}$; and two output levels distributed as follows as a function of the state of nature and the effort level:

	A	B
Probability	80%	20%
Output Under e_h	25	5
Output Under e_m	5	25
Output Under e_l	5	5

The principal is risk neutral, while the agent has a utility function $w^{1/2}$, when receiving a wage w, minus the effort cost which is zero if e_l is chosen, 1 if e_m is chosen, and 2 otherwise. The agent's reservation utility is 0.

(a) Derive the optimal wage schedule set by the principal when both effort and output are observable. [6 Marks]

Answer: If the effort chosen by the agent is observable, the principal would offer wages which depend only on the effort exerted and that always put the agent against his participation constraint:

$$\begin{aligned} w_h^{1/2} - 2 &= 0 \Rightarrow w_h = 4 \\ w_m^{1/2} - 1 &= 0 \Rightarrow w_m = 1 \\ w_l^{1/2} &= 0 \Rightarrow w_l = 0 \end{aligned}$$

At such wages the Agent would be indifferent between the three effort levels. Thus, he would choose e_h, the preferred option by the Principal, since:

$$\frac{4}{5}25 + \frac{1}{5}5 - 4 > \frac{1}{5}25 + \frac{4}{5}5 - 1 > 5$$
$$\text{since } 17 > 8 > 5.$$

Comments to grader: This they should do well.

(b) Derive the optimal wage schedule set by the principal when only output is observable. [10 Marks]

Answer: Begin by computing the optimal wage schedule offered by the Principal when he wants the Agent to exert effort e_h. Let the Principal offer wage $\overline{w}$ when output is high and wage $\underline{w}$ if output is low. The Principal thus, must post wages $\{\overline{w}, \underline{w}\}$ which guarantee that it is in the Agent's best interest to choose e_h. Thus, the IC constraint for the Agent requires:

$$\begin{aligned} U(e_h|\overline{w}, \underline{w}) = \frac{4}{5}\overline{w}^{1/2} + \frac{1}{5}\underline{w}^{1/2} - 2 &\geq \frac{1}{5}\overline{w}^{1/2} + \frac{4}{5}\underline{w}^{1/2} - 1 = U(e_m|\overline{w}, \underline{w}) \\ &\geq \underline{w}^{1/2} = U(e_l|\overline{w}, \underline{w}) \end{aligned}$$

Optimal wages are found by solving the participation constraint and the incentive constraint that compares effort e_h to effort e_l.

$$\frac{4}{5}\overline{w}^{1/2} + \frac{1}{5}\underline{w}^{1/2} - 2 = 0 \;\; \& \;\; \frac{4}{5}\overline{w}^{1/2} + \frac{1}{5}\underline{w}^{1/2} - 2 = \underline{w}^{1/2}$$
$$\frac{4}{5}\overline{w}^{1/2} + \frac{1}{5}\underline{w}^{1/2} - 2 > \frac{1}{5}\overline{w}^{1/2} + \frac{4}{5}\underline{w}^{1/2} - 1$$

Thus, $\overline{w} = 25/4$ and $\underline{w} = 0$. If the Principal wants to sustain the low effort e_l, the optimal wage, $w_* = 0$, is found instead, by solving the participation constraint at e_l:

$$w_*^{1/2} = 0$$

Finally there is no vector of posted wages $\{\overline{w}, \underline{w}\}$ which guarantees that it is in the Agent's best interest to choose e_m since

$$\begin{aligned}\frac{1}{5}\overline{w}^{1/2} + \frac{4}{5}\underline{w}^{1/2} - 1 &> \frac{4}{5}\overline{w}^{1/2} + \frac{1}{5}\underline{w}^{1/2} - 2 \;\Rightarrow\; \overline{w}^{1/2} - \underline{w}^{1/2} < \frac{5}{3} \\ \frac{1}{5}\overline{w}^{1/2} + \frac{4}{5}\underline{w}^{1/2} - 1 &> \underline{w}^{1/2} \;\Rightarrow\; \overline{w}^{1/2} - \underline{w}^{1/2} > 5.\end{aligned}$$

In this example the Principal prefers effort e_h to e_l since,

$$\frac{4}{5}(25 - \overline{w}) + \frac{1}{5}(5 - \underline{w}) = 16 > 5 = 5 - w_*.$$

The Principal cannot fully insure the Agent with incomplete information since it would undermine the incentives to exert effort. If effort were observable to the Principal, he would induce the Agent to exert high effort by fully insuring him. Since the Agent is risk averse, the profits of the Principal are smaller than in part (a) as the Principal must compensate the Agent for the wage volatility.

Comments to grader: Give partial credit, even if students may forget non-negativity.

(c) How much would the principal be willing to pay for a technology that reveals whether the agent chose e_h? [4 Marks]

Answer: As such technology suffices to replicate the complete information outcome the principal would be willing to pay at most 1 for such a technology, as his profits would increase from 16 to 17.