Summer 2012 Solutions

EC202

Microeconomic Principles II

2011/2012 Syllabus

Instructions to candidates
Time allowed: 3 hours + 10 minutes reading time.
This paper contains seven questions in three sections. Answer question one (section A) and **THREE** other questions, at least **ONE** from section B and at least **ONE** from section C. Question one carries 40% of the total marks; the other questions each carry 20% of the total marks.

Calculators are NOT permitted

Section A

1. Answer any **five** questions from (a)-(h). Each question carries eight marks.

   (a) State whether each of the following statements is true or false. Briefly explain your answers

   - "Long-run marginal cost must be less than or equal to short-run marginal cost."
   - "If an exchange economy is replicated a large number of times the core of the economy shrinks to a single allocation."
   - "Increasing relative risk aversion implies increasing absolute risk aversion."

   Answer:

   - FALSE. By definition long-run total cost is less than or equal to short-run total cost (where optimisation is subject to an extra constraint). So long-run *average* cost is less than or equal to short-run *average* cost. SRMC will be less than LRMC for some levels of output and greater at others. 2 mks
   - FALSE. The core shrinks to the set of competitive equilibria; there may be more than one CE. Extra credit for pointing out that replication must be balanced. 3 mks
   - ÁFALSE. ARA is $\alpha(x) := -u_{xx}(x)/u_x(x)$ and RRA is $\rho(x) := -xu_{xx}(x)/u_x(x) = x\alpha(x)$. So if $\alpha(x)$ is increasing, $\rho(x)$ must be increasing but not vice versa 3 mks

   (b) It is often claimed that the price system enables economic decision-making to be decentralised. Explain what this means and the conditions under which the price mechanism performs this role.

   Answer: The Robinson Crusoe economy (single agent both producing and consuming) could be used as a model to illustrate the point. If both the attainable set (production) A and the better-than-set (bounded by indifference curves) B are both convex then it is possible to find a set or prices $\mathbf{p}$ such that Crusoe's optimum is profit-maximising over A using $\mathbf{p}$ and expenditure-minimising over B using $\mathbf{p}$. 8 mks

   (c) State Walras' law. Explain it in terms of the properties of individual agents' demand and supply functions.

   Answer: In an n-good economy if $E_i(\mathbf{p})$ is excess demand for good i at any price vector $\mathbf{p}$ then Walras' Law requires that

$$\sum_{i=1}^{n} p_i E_i(\mathbf{p}) = 0.$$

   The result can be established using the adding-up property for each consumer's system of demand functions (including supply of factors). This in turn follows from the fact that each agent will be on the boundary of the budget set. 8 mks

(d) Under what circumstances is it possible to infer changes in social welfare from changes in national income?

Answer: If social welfare W is a function of individual utilities, if there are no externalities, if income is optimally distributed and if individuals act as utility-maximising price takers then ΔW is proportional to sum of changes in individual incomes. 8 mks

(e) Consider two competing lobbyists trying to influence a politician with donations. The politician has to choose between one of two policies $\{A, B\}$. The revenue of the first lobbyist is $x_A > 0$ if policy A is chosen and zero otherwise; while the revenue of the second lobbyist is $x_B > x_A$ if policy B is chosen and zero otherwise. Both lobbyists can decide how much spend to influence the politician's decision. The politician chooses the policy A whenever the first lobbyist spends strictly more than the second lobbyist, and policy B otherwise.

i. Set up the game played by the two lobbyists as a game of complete information. Derive the best response of one of the two lobbyists. **(5 Marks)**

Answer: The set of players in the game is $N = \{A, B\}$, the set of actions for each of the two players is $\mathbb{R}_+$, and for any expenditure profile $(e_A, e_B) \in \mathbb{R}^2_+$, the utility maps are:

$$u_A(e_A, e_B) = \begin{cases} x_A - e_A & \text{if } e_A > e_B \\ -e_A & \text{if } e_A \leq e_B \end{cases}$$

$$u_B(e_A, e_B) = \begin{cases} -e_B & \text{if } e_A > e_B \\ x_B - e_B & \text{if } e_A \leq e_B \end{cases}$$

The best responses of the two players are respectively determined by:

$$b_A(e_B) = \begin{cases} 0 & \text{if } x_A \leq e_B \\ e_B^+ & \text{if } x_A > e_B \end{cases}$$

$$b_B(e_A) = \begin{cases} 0 & \text{if } x_B < e_A \\ \{0, e_A\} & \text{if } x_B = e_A \\ e_A & \text{if } x_B > e_A \end{cases}$$

for e_B^+ the smallest possible number exceeding e_B. The best response of player A follows since:

$$u_A(0, e_B) = 0 \geq x_A \mathbb{I}(e_A > e_B) - e_A = u_B(e_A, e_B) \text{ if } e_B \geq x_A \text{ and } e_A > 0$$
$$u_A(e_B^+, e_B) = x_A - e_B^+ \geq x_A \mathbb{I}(e_A > e_B) - e_A = u_B(e_A, e_B) \text{ if } e_B < x_A \text{ and } e_A \neq e_B$$

while the best response of player B follows since:

$$u_B(0, e_A) = 0 \geq x_B \mathbb{I}(e_B \geq e_A) - e_A = u_B(e_B, e_A) \text{ if } e_A \geq x_B \text{ and } e_B > 0$$
$$u_B(e_A, e_A) = x_B - e_A \geq x_B \mathbb{I}(e_B \geq e_A) - e_A = u_B(e_B, e_A) \text{ if } e_A \leq x_B \text{ and } e_B \neq e_A$$

Comments to grader: Give partial credit. Best responses are worth 2 marks and their plot 1 mark. Derivation of best responses from inequalities should warrant an additional mark.

ii. Does the game possess a Pure Strategy Nash Equilibrium? Explain. **(3 Marks)**

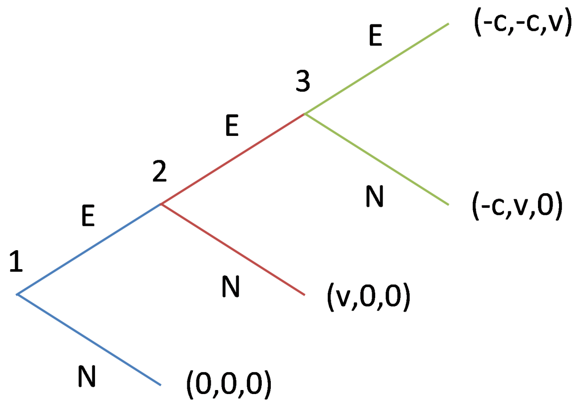
Answer: The game does not possesses any Pure Strategy Nash equilibrium. If e_B is a best response for B, it must be that $e_B \in \{0, e_A\}$ by part (ii). However, when $e_B = 0$, $b_A(e_B) = 0^+$. Thus this cannot be an equilibrium since $b_B(0^+) > 0$ as $x_B > 0^+$. If instead, $e_B = e_A > 0$, $b_A(e_B) \in \{0, e_B^+\}$. Thus this cannot be an equilibrium since $b_B(0) = 0$ and because $b_B(e_B^+) > e_b$ as $x_B > x_A > e_B$.

Comments to grader: Give partial credit. An argument must be provided to award full credit.

(f) Three hungry lions are members of a hierarchical group, and face a prey. If lion 1 does not eat the prey, the prey escapes, and the game ends. If it eats the prey, it becomes slow, and lion 2 can eat it. If lion 2 does not eat lion 1, the game ends; if it eats lion 1, it may then be eaten by lion 3. Each lion prefers to eat than to be hungry, but prefers to be hungry than to be eaten.

i. Draw the extensive form game. How could you map such a dynamic game to a static game of complete information? **(2 Marks)**

Answer: The extensive form game satisfies:



The strategic form representation of the game instead, satisfies:

3	E			N		
	$1 \backslash 2$	E	N	$1 \backslash 2$	E	N
	E	$-c,-c,\underline{v}$	$\underline{v},\underline{0},\underline{0}$	E	$-c,\underline{v},0$	$\underline{v},0,\underline{0}$
	N	$\underline{0},\underline{0},\underline{0}$	$0,\underline{0},\underline{0}$	N	$\underline{0},\underline{0},\underline{0}$	$0,\underline{0},\underline{0}$

ii. Find all the Pure Strategy Nash Equilibria of the game and the unique Subgame Perfect Equilibrium of the game. **(4 Marks)**

Answer: By the best responses underlined in the above matrix it is clear that three Pure Strategy Nash Equilibria exist in this game. Namely: (1) $a_1 = a_3 = E$ and $a_2 = N$; (2) $a_2 = a_3 = E$ and $a_1 = N$; (3) $a_1 = a_3 = N$ and $a_2 = E$. The only SPE satisfies: $a_1 = a_3 = E$ and $a_2 = N$. This is the case by backward induction since:

$$\begin{aligned} u_3(E) &= v > 0 = u_3(N) \\ u_2(N, E) &= 0 > -c = u_2(E, E) \\ u_1(E, N, E) &= v > 0 = u_1(N, N, E) \end{aligned}$$

Comments to grader: An argument, even a simple one, must be provided to award full credit. Checking for deviations on the table could suffice.

iii. What would be Subgame Perfect Equilibrium of the game if the group were composed by a finite number n of lions. **(2 Marks)**

Answer: The same principle of the previous answer applies. The last of the n lions certainly eats, and the preceding lions alternate between eating and not eating. In particular in the unique SPE

$$a_j = \begin{cases} E & \text{if} \quad j \in \{n, n-2, n-4, ...\} \\ N & \text{if} \quad j \in \{n-1, n-3, n-5, ...\} \end{cases}$$

This is a SPE since:

$$\begin{aligned} u_j(E, a_{-j}) &= v > 0 = u_j(N, a_{-j}) & \text{if } j \in \{n, n-2, n-4, ...\} \\ u_j(N, a_{-j}) &= 0 > -c = u_j(E, a_{-j}) & \text{if } j \in \{n-1, n-3, n-5, ...\} \end{aligned}$$

Comments to grader: Easy question throughout.

(g) Consider an economy with two goods: money m and consumption x. A single firm sells consumption good in the market. The firm uses money to produce consumption. In particular, it costs $x^{3/2}$ units of money to produce x units of consumption. All the consumers are identical. Every one of them is endowed with M units of money, but with no consumption. The preferences of a consumer with x units of consumption and m units of money satisfy:

$$U(x, m) = 3x^{1/2} + m$$

i. Find the profit maximizing price that the monopolist would set, if he is bound to choose among linear pricing schedules, $P(x) = px$. **(3 Marks)**

Answer: In this environment, any consumer chooses x in order to maximize his payoff, given the price schedule px:

$$\max_x U(x, y(x)) \quad \Rightarrow \quad \frac{3}{2}x^{-1/2} = p \qquad \text{(FOC)}$$

Thus, consumer demand is given by:

$$x^*(p) = \frac{9}{4p^2}$$

Given such demands, a firm chooses p to maximize profits:

$$\max_p px - x^{3/2} \quad \text{subject to FOC}$$

As usual, the firm can effectively choose x by changing p. Using this observation the problem of the monopolist simplifies to:

$$\max_x \frac{3}{2}x^{1/2} - x^{3/2} \quad \Rightarrow \quad \frac{3}{4}x^{-1/2} = \frac{3}{2}x^{1/2}$$

Desired equilibrium demand and price p therefore satisfy:

$$x^* = 1/2 \ \ \& \ \ p = 3/\sqrt{2}$$

Comments to grader: The consumer problem is worth 1 marks, and the firm problem is worth 2 marks. Please assign partial credit.

ii. Find the profit maximizing price schedule that the monopolist would set, if he is bound to choose among two-part tariffs, $P(x) = p_0 + p_1 x$. **(5 Marks)**
Answer: In this environment, the participation constraint of consumer requires:

$$U(x,y) - U(0,M) = 3x^{1/2} - P(x) \geq 0 \tag{PC}$$

If PC holds, a consumer chooses $x > 0$ in order to maximize his payoff, given the price schedule $P(x)$:

$$\max_x U(x, y(x)) \quad \Rightarrow \quad \frac{3}{2}x^{-1/2} = P'(x) = p_1$$

Thus, the demand by consumer facing a two-part tariff $P(x)$ is given by:

$$x^*(P) = \begin{cases} \dfrac{9}{4p_1^2} & \text{if } \; 3x^{1/2} - P(x) \geq 0 \\ 0 & \text{if } \; 3x^{1/2} - P(x) < 0 \end{cases}$$

Given such demands, a firm chooses P to maximize profits:

$$\max_P P(x) - x^{3/2} \;\text{ subject to PC and FOC}$$

As usual, when the fixed fee is chosen optimally, PC holds with equality at $x^*(P)$. Thus $P(x) = 3x^{1/2}$. The firm can effectively choose x by changing p_1. Using these two facts the problem of the monopolist simplifies to:

$$\max_x 3x^{1/2} - x^{3/2} \quad \Rightarrow \quad \frac{3}{2}x^{-1/2} = \frac{3}{2}x^{1/2}$$

where the latter implies that: $MRS = MRT$. The consumer's first order conditions further required that $MRS = p_1$. Desired equilibrium demand and marginal tariff p_1 therefore satisfy:

$$x^* = 1 \;\;\&\;\; p_1 = 3/2$$

The fixed fee of the optimal two-part tariff is then found by solving the participation constraint at the optimum:

$$P(x^*) = p_0 + p_1 x^* = 3(x^*)^{1/2}$$
$$p_0 = 3(x^*)^{1/2} - 1.5x^* = 1.5$$

Comments to grader: The consumer problem is worth 2 marks, and the firm problem is worth 3 marks. Please assign partial credit.

(h) Consider Spence's signalling model. A worker's type is $t \in \{1, 4\}$. The probability that any worker is of type $t = 1$ is equal to $1/4$, while the probability that $t = 4$ is equal to $3/4$. The productivity of a worker in a job is $t^{1/2}$. Each worker chooses a level of education $e \geq 0$. The total cost of obtaining education level e is $C(e|t) = e^2/t$. The worker's wage is equal to his expected productivity.

i. Find a pooling Perfect Bayesian Equilibrium. **(4 Marks)**
Answer: In a pooling PBE, the two types of workers choose the same education level, denote it by $\overline{e}$. It implies that the equilibrium posterior beliefs when $e = \overline{e}$ is observed are the same as the prior beliefs. Hence the wage paid, when $e = \overline{e}$, equals the expected (prior) productivity, namely:

$$(3/4)2 + (1/4)1 = 7/4$$

In the proposed pooling PBE, if an education level e is observed which is not equal to $\overline{e}$, then it is conjectured that the worker's type is low ability and hence wage equal to 1 is offered. Given this, the incentive compatibility conditions that ensure that neither type has an incentive to deviate are as follows. For the high productivity type, it must be that for any education level e:

$$7/4 - \overline{e}^2/4 \geq 1 - e^2/4 \quad \Leftrightarrow \quad 7 - \overline{e}^2 \geq 4$$

since the RHS is maximized at $e = 0$. Hence, high-type's IC is satisfied if and only if $\overline{e} \leq \sqrt{3}$.
For the low type, it must be that for any education level e:

$$7/4 - \overline{e}^2 \geq 1 - e^2 \quad \Leftrightarrow \quad 7/4 - \overline{e}^2 \geq 1$$

since the RHS is maximized at $e = 0$. Hence, low-type's IC is satisfied if and only if $\overline{e} \leq \sqrt{3}/2$.
In summary, both types' IC conditions are satisfied provided that $\overline{e} \leq \sqrt{3}/2$. Hence, there exists a continuum of pooling PBE in which both types choose the same, strictly positive education e; for any level e less than or equal to $\sqrt{3}/2$.
Comments to grader: Give partial credit. Students need only to find one of the equilibria discussed and to argue why it is an equilibrium. No need for complete derivation.

ii. Find a separating Perfect Bayesian Equilibrium. **(4 Marks)**
Answer: In a separating PBE, the two types of workers choose different education levels, e_H and e_L, where the former is the education level of the high type, with $t = 4$ and the former of the low type, $t = 1$. The equilibrium posterior beliefs are such that: (i) when $e = e_H$ is observed, the worker is of high type with probability one; (ii) when any other education level e is observed (including e_L), the worker is of low type with probability one. Hence, the wage paid when $e = e_H$ is observed equals 2, while it equal 1 when any other education level e is observed. For such beliefs, the incentive constraint of the high type requires him to prefer education e_H to any other $e \neq e_H$:

$$2 - e_H^2/4 \geq 1 - e^2/4 \quad \Leftrightarrow \quad 2 - e_H^2/4 \geq 1$$

where the implication holds since the RHS is maximized at $e = 0$. Hence, high-type's IC is satisfied if and only if $e_H \leq 2$.
The low type must also prefer not to deviate to any other $e \neq e_H$:

$$1 - e_L^2 \geq 1 - e^2 \quad \Leftrightarrow \quad 1 - e_L^2 \geq 1$$

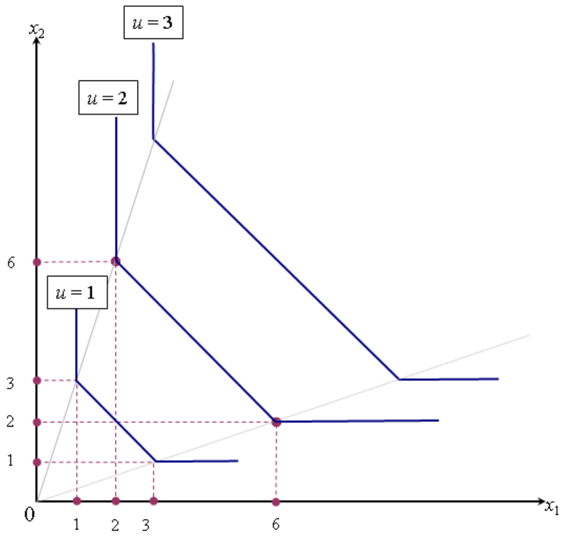
where the implication holds since the RHS is maximized at $e = 0$. Hence, low-type's IC is satisfied if and only if $e_L = 0$. Moreover we need to check that the low type has no incentive to mimic the high type. That is:

$$1 - e_L^2 \geq 2 - e_H^2 \quad \Leftrightarrow \quad e_H^2 \geq 1 + e_L^2$$

Since $e_L = 0$, the latter inequality simply requires that the high-type's IC is satisfied if and only if $e_H \geq 1$.
Therefore all the separating PBE are characterized by education levels $e_H \in [1, 2]$ and $e_L = 0$.
Comments to grader: Give partial credit. Students need only to find one of the equilibria discussed and to argue why it is an equilibrium. No need for complete derivation.

Section B [Answer at least ONE question]

2. Suppose a person's preferences are given by the indifference curves illustrated in the figure, where x_1, x_2 denote quantities of two goods and u denotes utility level.



(a) Do the preferences satisfy the axioms of (i) greed (ii) strict quasiconcavity (iii) smoothness? Briefly explain your answers.

(b) The person wants to find the minimum budget that will enable him to achieve a utility level of 2 when the prices of the goods are p_1 and p_2. Using the diagram describe the solution to this problem for the following cases: (i) $p_1 < p_2$, (ii) $p_1 > p_2$, (iii) $p_1 = p_2$.

(c) Provide a general definition of the consumer's cost function and find the cost function for these preferences.

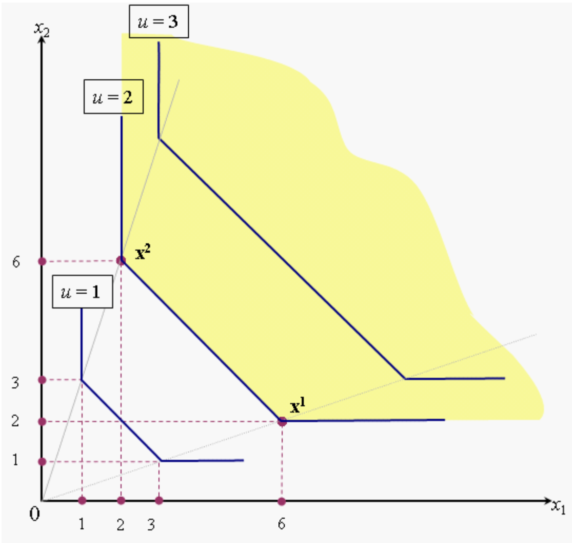
(d) Provide a general definition of the indirect utility function and, using the answer to part (c), find the indirect utility function for these preferences.

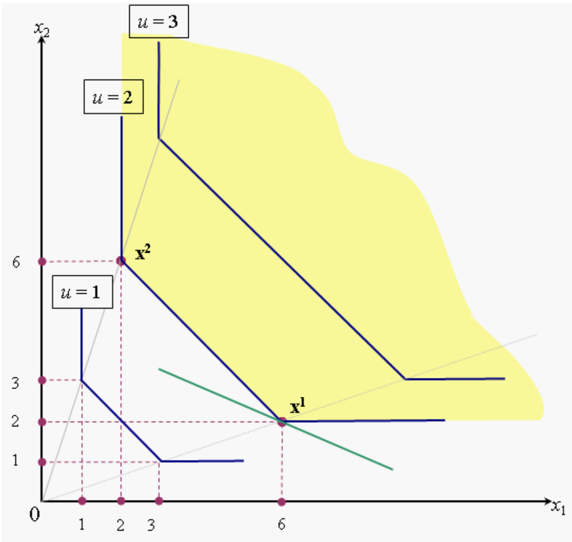
(e) Suppose prices are $p_1 = 1$, $p_2 = 2$. The government introduces a sales tax on good 1 so that the price to consumers increases. It also wants to compensate some consumers (all of whom have these preferences) so that they are no worse off. An adviser suggests that this can be done by increasing these consumers' incomes so that they consume exactly the bundle that they were able to consume before the tax was introduced. Is this advice correct?

Answer:

(a) Greed – no (there are some cases where if the quantity of at least one good increases utility does not increase). Quasiconcave but not strictly quasiconcave (flat sections in indifference curves). Smoothness – no (kinks). 4 mks

(b) If $p_1 < p_2$ then the person will minimise expenditure at bundle $\mathbf{x}^1$; If $p_1 > p_2$ then the person will minimise expenditure at $\mathbf{x}^2$. If $p_1 = p_2$ then the person will minimise expenditure anywhere on the line joining $\mathbf{x}^1$ and $\mathbf{x}^2$. 4 mks





(c) The cost (expenditure) function is the minimised expenditure, expressed as a function of prices and utility $C(\mathbf{p}, u)$. In this case, minimised expenditure is

$$\min \{3p_1 + p_2, p_1 + 3p_2\} \cdot u.$$

4 mks

(d) The indirect utility function is maximised utility, expressed as a function of prices and income $V(\mathbf{p}, y)$. Writing "income = expenditure" and using the above expenditure function we see that the indirect utility function is

$$\frac{y}{\min \{3p_1 + p_2, p_1 + 3p_2\}}$$

4 mks

(e) For a small tax yes. To see this note that, at these prices, cost minimisation for $u = 2$ is at point $\mathbf{x}^1$; increasing the price of good 1 a little will not change that (the price of good 1 would need to at least double to move away from this point). So the consumer just needs to be given enough income to be able to consume $\mathbf{x}^1$. 4 mks

3. A firm has the cost function $a_0 + a_1 q + a_2 q^2$ where q is output and a_0, a_1, a_2 are positive parameters.

(a) If the firm is a price-taker, find its optimal output level, for an exogenously given price p.

(b) If, instead, the firm is a monopolist, find the expression for the firm's marginal revenue in terms of output, assuming that market price is given by $p = b_1 + b_2 q$, where $b_1 > a_1$ and $b_2 < 0$. Illustrate the optimum in a diagram and show that the firm will produce

$$\hat{q} = \frac{b_1 - a_1}{2\left[a_2 - b_2\right]}.$$

What is the price charged $\hat{p}$ and the marginal cost $\hat{c}$ at this output level?

(c) Suppose this monopoly is regulated: the regulator can control the price by setting a ceiling $p_{\max}$. Plot the average and marginal revenue curves that would then face the monopolist.

(d) Show that if the price ceiling is set so that $\hat{c} < p_{\max} < \hat{p}$ then the firm's output will rise above $\hat{q}$.

(e) What will happen if $p_{\max}$ does not satisfy this condition?

Answer:

(a) Average costs are

$$\frac{a_0}{q} + a_1 + a_2 q$$

which are a minimum at

$$\underline{q} = \sqrt{\frac{a_0}{a_2}} \tag{1}$$

where average costs are

$$2\sqrt{a_0 a_2} + a_1 \tag{2}$$

Marginal and average costs are illustrated in Figure 1. For a price above the level (2) the first-order condition for maximum profits is given by

$$p = a_1 + 2a_2 q$$

from which we find the supply curve as

$$q = \begin{cases} \frac{p-a_1}{2a_2} & \text{if } p \geq 2\sqrt{a_0 a_2} + a_1 \\ 0 & \text{otherwise} \end{cases}$$

– see Figure 1. 4 mks

(b) If the firm is a monopolist marginal revenue is

$$\frac{\partial}{\partial q}\left[b_1 q + b_2 q^2\right] = b_1 + 2b_2 q$$

Hence the first-order condition for the monopolist is

$$b_1 + 2b_2 q = a_1 + 2a_2 q \tag{3}$$

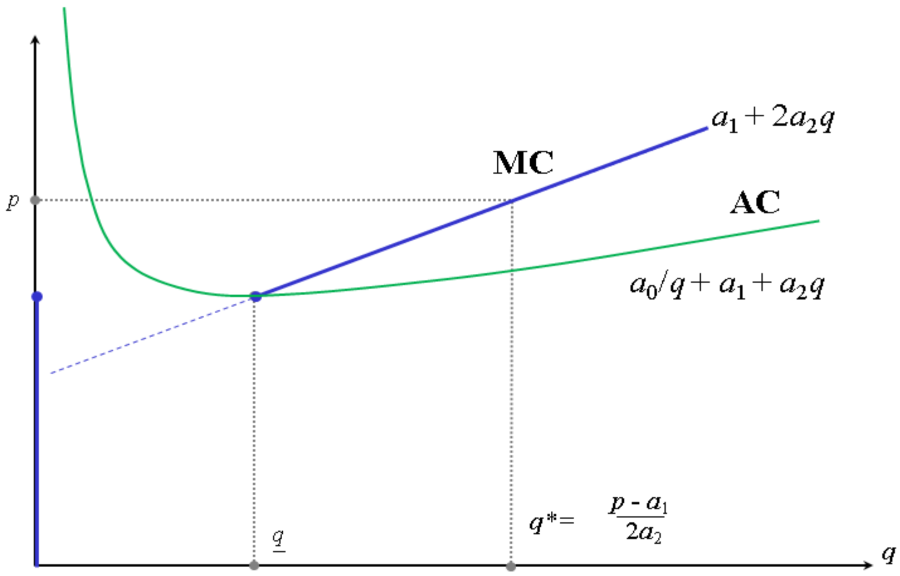


Figure 1: Supply curve for competitive case

from which the solution $\hat{q}$ follows. Substituting for $\hat{q}$ we also get

$$\hat{c} = b_1 + 2b_2\hat{q} = \frac{a_2b_1 - a_1b_2}{a_2 - b_2} \tag{4}$$

$$\hat{p} = b_1 + b_2\hat{q} = b_1 + b_2\frac{b_1 - a_1}{2\left[a_2 - b_2\right]} \tag{5}$$

– see Figure 2. 5 mks

(c) Consider how the introduction of a price ceiling will affect average revenue. Clearly we now have

$$\text{AR}(q) = \left\{ \begin{array}{l} p_{\max} \text{ if } q \leq q_0 \\ b_1 + b_2q \text{ if } q \geq q_0 \end{array} \right\} \tag{6}$$

where $q_0 := \left[p_{\max} - b_1\right]/b_2$: average revenue is a continuous function of q but has a kink at q_0. From this we may derive marginal revenue which is

$$\text{MR}(q) = \left\{ \begin{array}{l} p_{\max} \text{ if } q < q_0 \\ b_1 + 2b_2q \text{ if } q > q_0 \end{array} \right\} \tag{7}$$

– notice that there is a discontinuity exactly at q_0. The modified curves (6) and (7) are shown in Figure 3: notice that they coincide in the flat section to the left of q_0. Clearly the outcome depends crucially on whether MC intersects (modified) MR (i) to the left of $\hat{q}$, (ii) to the right of q_0, (iii) between $\hat{q}$ and q_0, or in the discontinuity exactly at q_0 4 mks

(d) Case (iii) is the relevant one, and it is clear that output will have risen from $\hat{q}$ to q_0. The other cases can easily be found by appropriately shifting the curves on the diagram. 4 mks

(e) We can use the above diagram to show that if $p_{\max} < \hat{c}$ then output falls; if $p_{\max} > \hat{p}$ output remains unchanged. 3 mks

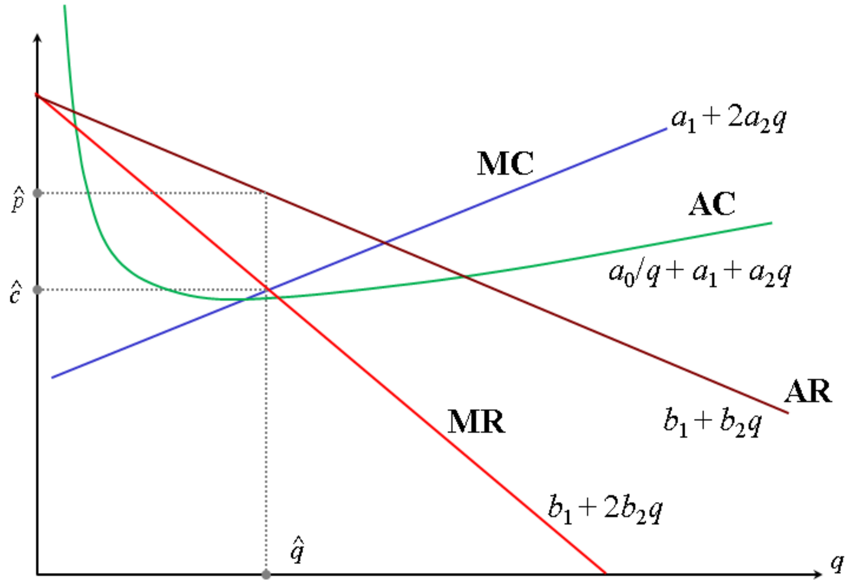


Figure 2: Unregulated Monopoly

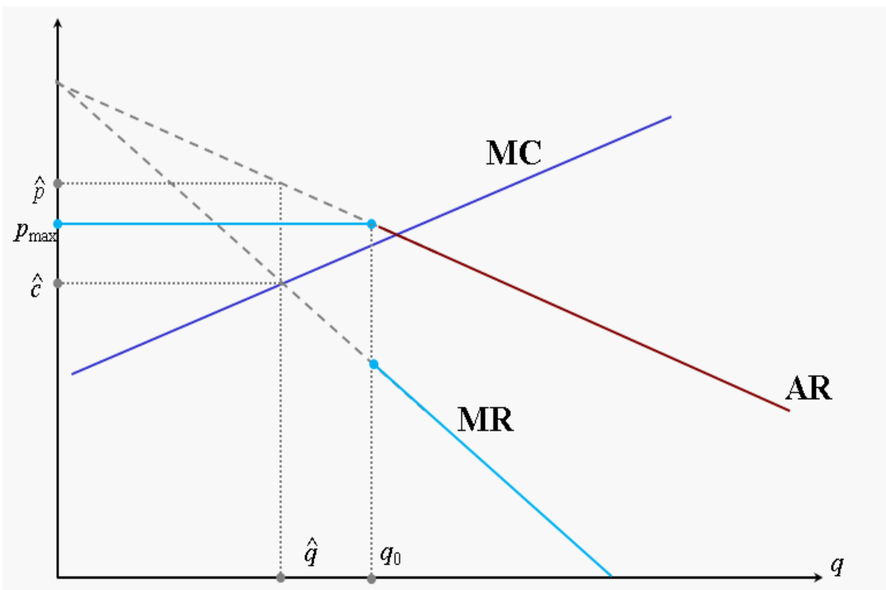


Figure 3: The solution for Regulated Monopoly:

4. A person with wealth y_0 is considering investing in a risky enterprise. If the enterprise succeeds the value of the investment will double; if it fails everything invested is lost.

(a) If the person invests x and the probability of success is $\pi > \frac{1}{2}$ what is

- ex-post wealth y in the case of success?
- ex-post wealth in the case of failure?
- expected ex-post wealth $\mathbb{E}y$?

(b) If the person's utility is given by $\mathbb{E}\log(y)$, find the optimal size of investment x.

(c) The government proposes to tax any *gain* from the investment at a rate t but without a loss-offset provision: the payoff in the case of success would be $[1 - t]x$ but the outcome in the case of failure would be just as before. Find the optimal x.

(d) Suppose that the government were to modify this tax and allow full loss offset, so that in the case of failure one only loses $[1 - t]x$ rather than x. Again find the optimal x.

(e) Suppose the government abandoned the proposed tax on gains and replaced it with a tax on *ex-post wealth*. Show that the investment decision would be exactly the same as in part (b). Why is this?

(f) Show that a rise in the tax rate would reduce investment under tax scheme (c), increase it under tax scheme (d) and leave it unchanged under tax scheme (e).

Answer:

(a) In the case of success the investor would have $[y_0 - x] + 2x = y_0 + x$ as ex-post wealth. Therefore

$$y = \begin{cases} y_0 + x & \text{with probability} \quad \pi \\ \\ y_0 - x & \text{with probability} \quad 1 - \pi \end{cases}.$$

Expected ex-post wealth is $\mathbb{E}y = \pi [y_0 + x] + [1 - \pi] [y_0 - x] = y_0 + x [2\pi - 1]$ 2 mks

(b) The value of x is chosen to maximise

$$\pi \log(y_0 + x) + [1 - \pi] \log(y_0 - x).$$

Assume that there is an interior solution. The first-order condition for a maximum is

$$\frac{\pi}{y_0 + x} + \frac{\pi - 1}{y_0 - x} = 0 \qquad (8)$$

which implies

$$x^* = [2\pi - 1] y_0 \qquad (9)$$

which is positive given the condition $\pi > \frac{1}{2}$ specified in the question. 4 mks

(c) Without loss offset final wealth would be

$$y = \begin{cases} y_0 + x [1 - t] & \text{with probability} \quad \pi \\ \\ y_0 - x & \text{with probability} \quad 1 - \pi \end{cases}.$$

So the value of x is chosen to maximise

$$\pi \log(y_0 + x[1-t]) + [1-\pi]\log(y_0 - x).$$

Again assume that there is an interior solution. The first-order condition for a maximum is now

$$\frac{\pi[1-t]}{y_0 + x[1-t]} + \frac{\pi - 1}{y_0 - x} = 0.$$

Solving for x we find

$$x^{**} = \pi y_0 - \frac{1-\pi}{1-t} y_0 \tag{10}$$

Clearly this only makes sense if $x^{**} \geq 0$ which requires

$$t \leq \frac{2\pi - 1}{\pi}$$

4 mks

(d) With full loss offset the final wealth is

$$y = \begin{cases} y_0 + x[1-t] & \text{with probability} \quad \pi \\ y_0 - x[1-t] & \text{with probability} \quad 1-\pi \end{cases}.$$

The first-order condition for an interior maximum is

$$\frac{\pi - 1}{y_0 - x[1-t]} + \frac{\pi}{y_0 + x[1-t]} = 0$$

and so we have

$$x^{***} = \frac{2\pi - 1}{1-t} y_0. \tag{11}$$

4 mks

(e) Now

$$y = \begin{cases} [y_0 + x][1-t] & \text{with probability} \quad \pi \\ [y_0 - x][1-t] & \text{with probability} \quad 1-\pi \end{cases}$$

Following the same argument as before, the FOC is

$$\frac{\pi - 1}{[y_0 - x][1-t]} + \frac{\pi}{[y_0 + x][1-t]} = 0$$

but this is exactly the same as in (8) so the solution must be the same and is given by (9). The reason for this is that preferences exhibit constant relative risk aversion – homothetic contours in state space. For any x taxing ex-post wealth just moves the outcome point in along a ray through the origin. 3 mks

(f) Differentiating (9), (10) and (11) we find

$$\begin{aligned} \frac{\partial x^*}{\partial t} &= 0 \\ \frac{\partial x^{**}}{\partial t} &= -\frac{1-\pi}{[1-t]^2} y_0 < 0 \\ \frac{\partial x^{***}}{\partial t} &= \frac{x^{***}}{1-t} > 0 \end{aligned}$$

3 mks

Section C [Answer at least ONE question]

5. Consider three board members $\{1, 2, 3\}$ choosing which of three candidates $\{A, B, C\}$ is to be appointed as the new CEO of a firm. The preferences of the three board members over candidates respectively satisfy:

$$u_1(A) = 1 > u_1(B) = x > u_1(C) = 0$$
$$u_2(B) = 1 > u_2(C) = y > u_2(A) = 0$$
$$u_3(C) = 1 > u_3(B) = z > u_3(A) = 0$$

The preferences of each board member are known by all the other members.

(a) First consider a scenario in which board members can secretly cast one vote in favor of either of the three candidates, and in which the candidate appointed is the one to receives more votes. If there is no outright winner, and two or more candidates receive the largest number of votes, the CEO is chosen at random among these candidates. Model the environment as a game of complete information. Which candidates can be appointed outright (i.e. without a tie) in a Pure Strategy Nash Equilibrium of this game? Find the corresponding strategies for each of these equilibria you discuss. Which candidates, if any, cannot be appointed in any Pure Strategy Nash Equilibrium of this game? **(7 Marks)**

Answer: This is a game among three players $N = \{1, 2, 3\}$. For any player $i \in N$ action sets A_i and preferences U_i satisfy:

$$A_i = \{A, B, C\}$$
$$U_i(a_1, a_2, a_3) = \begin{cases} u_i(a) & \text{if } a_k = a_j = a \text{ for some } j, k \in \{1, 2, 3\} \\ \sum_{a \in A_i} \frac{u_i(a)}{3} & \text{if } a_1 \neq a_2 \neq a_3 \end{cases}$$

Any one of the three candidates can be appointed outright in a PNE of the game. Namely, suppose that all the three vote members vote for candidate $a \in \{A, B, C\}$. So that the strategy satisfies $a_i = a$ for any board member i. If so, no board member benefits from a deviation since:

$$U_i(a, a, a) = u_i(a) \geq u_i(a) = U_i(b, a, a) \text{ for any } b \in A_i \text{ and } i \in \{1, 2, 3\}$$

All candidates can thus be appointed as a CEO in a PNE, since voters know that their vote cannot affect the outcome of the election, when the other two board members vote for a given candidate.

(b) Next consider a scenario in which board members can secretly cast one vote against either of the three candidates, and in which the candidate appointed is the one who receives fewer votes. If two or more candidates receive the smallest number of votes, the CEO is chosen at random among these candidates. Model the environment as a game of complete information. Which candidates can be appointed outright in a Pure Strategy Nash Equilibrium of this game, when $x, y, z > 1/2$? Find the corresponding strategies for every equilibrium that you discuss. Which candidates, if any, cannot be elected in any Pure Strategy Nash Equilibrium of this game? Explain. How would your answer change were $x, y, z < 1/2$. **(10 Marks)**

Answer: This is a game among three players $N = \{1, 2, 3\}$. For any player $i \in N$ action sets A_i and preferences U_i satisfy:

$$\begin{aligned} A_i &= \{A, B, C\} \\ U_i(a_1, a_2, a_3) &= \begin{cases} u_i(a) & \text{if } \; a_k \neq a \text{ for } \forall k \in N \text{ and } a_k \neq a_j \text{ for } j \in N \\ \sum_{a \in A_i \setminus b} \frac{u_i(a)}{2} & \text{if } \; a_1 = a_2 = a_3 = b \\ \sum_{a \in A_i} \frac{u_i(a)}{3} & \text{if } \; a_1 \neq a_2 \neq a_3 \end{cases} \end{aligned}$$

When $x, y, z > 1/2$, only candidates B can be appointed outright in a PNE of the game. All the PNE strategy profiles for this case are (a_1, a_2, a_3) are: (C, C, A), (A, C, A), (C, A, A), (C, A, C), (A, C, C). These equilibria can be found by looking the best responses underlined in the following strategic form representation of the game.

3	1\2	A	B	C
A	A	$\frac{x}{2}, \frac{1+y}{2}, \frac{1+z}{2}$	$0, y, \underline{1}$	$\underline{x}, \underline{1}, \underline{z}$
	B	$0, \underline{y}, \underline{1}$	$0, \underline{y}, \underline{1}$	$\frac{1+x}{3}, \frac{1+y}{3}, \underline{\frac{1+z}{3}}$
	C	$\underline{x}, \underline{1}, \underline{z}$	$\underline{\frac{1+x}{3}}, \frac{1+y}{3}, \underline{\frac{1+z}{3}}$	$\underline{x}, \underline{1}, \underline{z}$
B	A	$0, \underline{y}, \underline{1}$	$0, \underline{y}, \underline{1}$	$\frac{1+x}{3}, \frac{1+y}{3}, \frac{1+z}{3}$
	B	$0, \underline{y}, \underline{1}$	$\frac{1}{2}, \frac{y}{2}, \frac{1}{2}$	$\underline{1}, 0, 0$
	C	$\underline{\frac{1+x}{3}}, \underline{\frac{1+y}{3}}, \frac{1+z}{3}$	$\underline{1}, 0, 0$	$\underline{1}, 0, 0$
C	A	$\underline{x}, \underline{1}, z$	$\frac{1+x}{3}, \frac{1+y}{3}, \frac{1+z}{3}$	$\underline{x}, \underline{1}, \underline{z}$
	B	$\frac{1+x}{3}, \underline{\frac{1+y}{3}}, \frac{1+z}{3}$	$\underline{1}, 0, 0$	$\underline{1}, 0, 0$
	C	$\underline{x}, \underline{1}, \underline{z}$	$\underline{1}, 0, 0$	$\frac{1+x}{2}, \frac{1}{2}, \frac{z}{2}$

When $x, y, z < 1/2$, no candidates can be appointed outright in a PNE of the game. All the PNE strategy profiles for this case are (a_1, a_2, a_3) are: (B, C, A), (C, A, B). In both equilibria all the candidates tie. These equilibria can be found by looking the best responses underlined in the following strategic form representation of the game.

3	1\2	A	B	C
A	A	$\frac{x}{2}, \frac{1+y}{2}, \frac{1+z}{2}$	$0, y, \underline{1}$	$x, \underline{1}, z$
	B	$0, y, \underline{1}$	$0, y, \underline{1}$	$\underline{\frac{1+x}{3}}, \underline{\frac{1+y}{3}}, \underline{\frac{1+z}{3}}$
	C	$\underline{x}, \underline{1}, z$	$\underline{\frac{1+x}{3}}, \frac{1+y}{3}, \underline{\frac{1+z}{3}}$	$x, \underline{1}, \underline{z}$
B	A	$0, y, \underline{1}$	$0, y, \underline{1}$	$\frac{1+x}{3}, \underline{\frac{1+y}{3}}, \underline{\frac{1+z}{3}}$
	B	$0, \underline{y}, \underline{1}$	$\frac{1}{2}, \frac{y}{2}, \frac{1}{2}$	$\underline{1}, 0, 0$
	C	$\underline{\frac{1+x}{3}}, \underline{\frac{1+y}{3}}, \underline{\frac{1+z}{3}}$	$\underline{1}, 0, 0$	$\underline{1}, 0, 0$
C	A	$x, \underline{1}, z$	$\frac{1+x}{3}, \frac{1+y}{3}, \frac{1+z}{3}$	$x, \underline{1}, z$
	B	$\underline{\frac{1+x}{3}}, \underline{\frac{1+y}{3}}, \frac{1+z}{3}$	$\underline{1}, 0, 0$	$\underline{1}, 0, 0$
	C	$x, \underline{1}, z$	$\underline{1}, 0, 0$	$\frac{1+x}{2}, \frac{1}{2}, \frac{z}{2}$

Comments to grader: The argument provided here is long and dull. I report tables only in your interest as many students will choose this road. However, no need for tables to find the PNE. Please give full credit if students use more concise proofs. Easy, but long question.

(c) Consider an egalitarian social planner which maximizes the sum of utilities of the three board members. Which of the two voting rules described in part (a) and (b) would the social planner if he wanted to maximizes the sum of utilities in the least efficient Nash Equilibrium? **(3 Marks)**

Answer: Such a social planner would want to appoint B if $x + z > y$, and C vice versa. The planner would thus, always choose mechanism always (b). The worst equilibrium of part (a) appoints candidate A with probability 1, which is certainly worse than appointing, either B with certainty, or one between A, B, and C with equal probability.

6. A group of neighbors are choosing whether to contribute a fixed amount towards the provision of a pure public good that will benefit them all equally. The good is provided if and only if at least three of them contribute; if it is not provided, contributions are not refunded. Let v denote the value of the public good to each of the neighbors, let f denote the fixed contribution, and assume that $v > f > 0$.

(a) First assume that the group is composed of five neighbors. Find all pure strategy Nash equilibria of the game. Is there a Nash equilibrium in which more than 3 people contribute? One in which fewer than 3 contribute? Explain your reasoning carefully. **(7 Marks)**

Answer: Let $N = \{1, ..., 5\}$ denote the set of players in the game, and let $A_i = \{0, 1\}$ denote the action set for any player $i \in N$ (where 0 denotes no contribution and 1 a contribution). For any action profile $a \in \{0,1\}^5$, the utility of player i is given by:

$$u_i(a) = v\mathbb{I}\left(\sum_{j \in N} a_j \geq 3\right) - fa_i$$

The game possesses two types of pure strategy Nash equilibria: in one type nobody contributes, while in the other exactly three players contribute. In particular, the strategy profile $\overline{a} = (0, 0, 0, 0, 0)$ is a pure strategy Nash equilibrium since for any player $i \in N$:

$$u_i(\overline{a}) = 0 > -f = u_i(1, \overline{a}_{-i})$$

Similarly, any strategy profile $a' = (1, 1, 1, 0, 0)$ is a pure strategy Nash equilibrium since:

$$\begin{aligned} u_i(a') &= v - f > 0 = u_i(0, a'_{-i}) \text{ for } i \in \{1, 2, 3\} \\ u_i(a') &= v > v - f = u_i(1, a'_{-i}) \text{ for } i \in \{4, 5\} \end{aligned}$$

This holds clearly for any permutation of the strategy profile. A strategy profile cannot be a Nash equilibrium if only one or two players contribute since the players who contribute would prefer not to spend their resources on unrealized projects, as $-f < 0$. Similarly, a strategy profile cannot be a Nash equilibrium if more than three players contribute, since a players would prefer not to spend their resources on a project that is realized even without their contribution, as $v - f < v$.

(b) Next suppose that players face the same problem, but that they do not know whether there are three or five neighbors in the group. In particular, assume that, when choosing their contribution, players believe that the probability that there are only two more individuals is $q > 0$, while the probability that there are four more individuals is $1 - q > 0$. For which values of q does the game possess a Pure Strategy Bayes Nash equilibrium in which everyone contributes? For which values of q does the game possess a Pure Strategy Bayes Nash equilibrium in which nobody contributes? **(7 Marks)**

Answer: Since players do not observe whether only two or four other players are present when they participate in the game, strategies cannot condition depend on any type of information. For any action profile $a \in \{0,1\}^5$, the expected utility of player i is given by:

$$U_i(a) = v\left[q\mathbb{I}\left(\sum_{j \in N_3} a_j \geq 3\right) + (1-q)\mathbb{I}\left(\sum_{j \in N} a_j \geq 3\right)\right] - fa_i$$

where N_3 is a subset of N which includes player i. As before, the game possesses two types of pure strategy Bayes Nash equilibria: in one type nobody contributes, while in the other all contribute. In particular, the strategy profile $\overline{a} = (0,0,0,0,0)$ is a pure strategy Bayes Nash equilibrium since for any player $i \in N$:

$$U_i(\overline{a}) = 0 > -f = U_i(1, \overline{a}_{-i})$$

such equilibrium exist for any value of q. Any strategy profile $a' = (1,1,1,1,1)$ is a pure strategy Bayes Nash equilibrium when $q \geq f/v$ since for any player $i \in N$:

$$U_i(a') = v - f \geq (1-q)v = U_i(0, a'_{-i})$$

(c) Finally suppose that players face the same problem, but that only three neighbors $\{1,2,3\}$ are known to belong to the group, while the remaining two players $\{4,5\}$ may or may not be part of the group. In particular, assume that the prior probability that there are only players $\{1,2,3\}$ in the group is $q > 0$, while the probability that all players are present is $1-q > 0$. How would your answer to part (b) change in such an environment. **(6 Marks)**

Answer: In this setup, players $\{4,5\}$ know that if they are present there must be five players, while $\{1,2,3\}$ do not have any private information. For any action profile $a \in \{0,1\}^5$, the expected utility of player i is given by:

$$U_i(a) = \begin{cases} v\left[q\mathbb{I}\left(\sum_{j\in\{1,2,3\}} a_j \geq 3\right) + (1-q)\mathbb{I}\left(\sum_{j\in N} a_j \geq 3\right)\right] - fa_i & \text{if } \; i \in \{1,2,3\} \\ v\mathbb{I}\left(\sum_{j\in N} a_j \geq 3\right) - fa_i & \text{if } \; i \in \{4,5\} \end{cases}$$

As before, the game possesses two types of pure strategy Bayes Nash equilibria: in one type nobody contributes, while in the other only players $\{1,2,3\}$ contribute. In particular, the strategy profile $\overline{a} = (0,0,0,0,0)$ is a pure strategy Bayes Nash equilibrium since for any player $i \in N$:

$$U_i(\overline{a}) = 0 > -f = U_i(1, \overline{a}_{-i})$$

such equilibrium exist for any value of q. Any strategy profile $a' = (1,1,1,0,0)$ is a pure strategy Bayes Nash equilibrium when $q \geq f/v$ since for any player $i \in N$:

$$\begin{aligned} U_i(a') &= v - f > 0 = U_i(0, a'_{-i}) \text{ for } i \in \{1,2,3\} \\ U_i(a') &= v > v - f = U_i(1, a'_{-i}) \text{ for } i \in \{4,5\} \end{aligned}$$

Observe that having all players contribute is no longer a Bayes Nash equilibrium as players 4 and 5 would always benefit from the deviation as they could save the contribution and still enjoy the public good.

7. Consider a Principal-Agent problem with: three exogenous states of nature $\{H, M, L\}$; two effort levels $\{e_a, e_b\}$; and two output levels distributed as follows as a function of the state of nature and the effort level:

	H	M	L
Probability	20%	60%	20%
Output Under e_a	25	25	4
Output Under e_b	25	4	4

The principal is risk neutral, while the agent has a utility function $w^{1/2}$, when receiving a wage w, minus the effort cost which is zero if e_b is chosen, and 1 otherwise. The agent's reservation utility is 0.

(a) Derive the optimal wage schedule set by the principal when both effort and output are observable. **(6 Marks)**

Answer: If the effort chosen by the agent is observable, the Principal would offer wages which depend only on the effort exerted and that always put the Agent against his participation constraint:

$$\begin{aligned} w_a^{1/2} - 1 &= 0 \Rightarrow w_a = 1 \\ w_b^{1/2} &= 0 \Rightarrow w_b = 0 \end{aligned}$$

At such wages the Agent would be indifferent between the two effort levels. Thus, he would choose e_a, the preferred option by the Principal, since:

$$\begin{aligned} \frac{4}{5}25 + \frac{1}{5}4 - 1 &> \frac{1}{5}25 + \frac{4}{5}4 \\ 19.8 &> 8.2 \end{aligned}$$

Comments to grader: This they should do well.

(b) Derive the optimal wage schedule set by the principal when only output is observable. **(9 Marks)**

Answer: Begin by computing the optimal wage schedule offered by the Principal when he wants the Agent to exert effort e_a. Let the Principal offer wage $\overline{w}$ when output is high and wage $\underline{w}$ if output is low. The Principal thus, must post wages $\{\overline{w}, \underline{w}\}$ which guarantee that it is in the Agent's best interest to choose e_a. Thus, the IC constraint for the Agent requires:

$$U(e_a|\overline{w}, \underline{w}) = \frac{4}{5}\overline{w}^{1/2} + \frac{1}{5}\underline{w}^{1/2} - 1 \geq \frac{1}{5}\overline{w}^{1/2} + \frac{4}{5}\underline{w}^{1/2} = U(e_b|\overline{w}, \underline{w})$$

Optimal wages are found by solving the participation constraint and the incentive constraint at effort e_a. However only the incentive constraint will bind since wages are always non-negative, $\underline{w} \geq 0$.

$$\begin{aligned} \frac{4}{5}\overline{w}^{1/2} + \frac{1}{5}\underline{w}^{1/2} - 1 &\geq 0 \;\; \& \;\; \underline{w} = 0 \\ \frac{4}{5}\overline{w}^{1/2} + \frac{1}{5}\underline{w}^{1/2} - 1 &= \frac{1}{5}\overline{w}^{1/2} + \frac{4}{5}\underline{w}^{1/2} \end{aligned}$$

Thus, $\overline{w} = 25/9$ and $\underline{w} = 0$. If the Principal wants to sustain the low effort e_b, the optimal wage, $w_* = 0$, is found instead, by solving the participation constraint at e_b:

$$w_*^{1/2} = 0$$

In this example the Principal prefers effort e_a since:

$$\frac{4}{5}(25 - \overline{w}) + \frac{1}{5}(4 - \underline{w}) > \frac{1}{5}25 + \frac{4}{5}4 - w_*$$
$$18.6 > 8.2$$

The Principal cannot fully insure the Agent with incomplete information since it would undermine the incentives to exert effort. If effort were observable to the Principal, he would induce the Agent to exert high effort by fully insuring him. Since the Agent is risk averse, the profits of the Principal are smaller than in part (a) as the Principal must compensate the Agent for the wage volatility.

Comments to grader: Give partial credit, even if students may forget non-negativity.

(c) If the principal cannot observe effort, how much would he be willing to pay for a technology that, prior to the beginning of the game, reveals when the state L is realized. **(5 Marks)**

Answer: With such technology the principal would be able to pay the agent three different wages: $\mathring{w}$ when the state is L; $\overline{w}$ when output is high and the state is not L; and wage $\underline{w}$ otherwise. In the state L, the principal would not require the agent to exert high effort. The principal would set a wage $\mathring{w} = 0$ to induce effort $\mathring{e} = e_b$ at the agent's participation constraint in this state. If the state L were not realized instead, the incentive constraint for the Agent to exert high effort e_a would require:

$$\begin{aligned} & \bar{U}(e_a|\overline{w}, \underline{w}) = \overline{w}^{1/2} - 1 && \geq \frac{1}{4}\overline{w}^{1/2} + \frac{3}{4}\underline{w}^{1/2} = \bar{U}(e_b|\overline{w}, \underline{w}) \\ \iff \quad & 3\overline{w}^{1/2} - 3\underline{w}^{1/2} && \geq 4 \end{aligned}$$

In this state, optimal wages would then found by solving the participation constraint and the incentive constraint at effort e_a. However, only the incentive constraint would bind since wages are always non-negative, $\underline{w} \geq 0$.

$$\overline{w}^{1/2} - 1 \geq 0 \;\;\&\;\; \underline{w} = 0$$
$$3\overline{w}^{1/2} - 3\underline{w}^{1/2} = 4$$

Thus, $\overline{w} = 16/9$ and $\underline{w} = 0$. If the Principal instead, wanted to sustain effort e_b, the optimal wage $w_* = 0$ would be, as before, found by solving the participation constraint at e_b:

$$w_*^{1/2} = 0$$

The Principal would thus, always prefer effort e_a, when he know the state not to be L, since:

$$25 - \overline{w} > \frac{1}{4}25 + \frac{3}{4}4 - w_*$$
$$23.22 > 9.25$$

With such technology, the ex-ante profits of the principal are then given by:

$$\frac{4}{5}23.22 + \frac{1}{5}4 = 19.376$$

Thus, the principal would be willing to pay at most $19.376 - 18.6 = 0.776$ for such technology.

Comments to grader: Give partial credit. Students will find this to be hard.