Risk-sharing CARA individuals are collectively EU

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Abstract

This paper analyses the agreed choices, from uncertain financial prospects, of a group of individuals. The group’s agreed choices will conform to Expected Utility theory, if each individual has constant absolute risk-aversion, and if they share risk efficiently. © 1998 Elsevier Science S.A.

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1. Introduction

Consider a group of individuals jointly choosing between uncertain financial prospects. Their choice is joint in two senses. Firstly, the selection of any prospect must be unanimously agreed. Secondly, the (uncertain) outcomes of each prospect are sums of money, to be divided between members of the group, again according to unanimous prior agreement. An example of this situation might be a syndicate formed for investment, gambling or criminal purposes.

In this paper we are interested primarily in the consistency of such choices. Specifically, will the group’s agreed choices, from various hypothetical sets of alternative financial prospects, conform to the standard axioms of Expected Utility (EU) theory? In general, it turns out they will not do so. However, in the special case where each group member has Constant Absolute Risk Aversion (CARA), then consistency of this kind follows directly from the assumption of efficient risk-sharing.

2. Definitions and assumptions

A group of $m$ individuals has to agree a choice from a set of prospects typified by:

$$L = [X_1, \ldots, X_n; p_1, \ldots, p_n].$$

(1)

This prospect has a number of states $s = 1, 2, \ldots, n$, each with a probability $p_s$ and each resulting in a monetary prize of value $X_s$. The group has also to agree on a contingent distribution, among group

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members, of each possible prize in the chosen prospect. That is, it has to agree a \((m \times n)\) matrix \([x_{is}]\) such that:

\[
\forall s: \sum_{i=1}^{m} x_{is} = X_i, \tag{2}
\]

We assume no restriction on the sign of either \(X_i\) or \(x_{is}\).

The requirement of unanimous agreement, over both the selection of prospect and the contingent distribution of its monetary payoffs, defines this as a bargaining problem. As such, there are various approaches that could be taken to its analysis. Here, however, we assume only that the agreed prospect/distribution is \textit{ex ante efficient}. That is, there must not be any other feasible contingent distribution, of any available prospect, which every group member prefers, \textit{ex ante}, to that agreed.

We assume that each individual has a vNM utility function \(u(.)\) defined over his own final allocation of prize money. Thus, for any given prospect \(L\), individual \(i\)’s preferences over contingent distributions such as \([x_{is}]\) are representable by:

\[
v_i = \sum_{s=1}^{n} p_s u_i(x_{is}). \tag{3}
\]

Relative to other feasible contingent distributions of \(L\), \([x_{is}]\) is efficient if and only if there is no other such distribution preferred by every group member. A necessary condition for this is that there exist nonnegative \(\lambda_i (i = 1, 2, \ldots, m)\) and \(\mu_s (s = 1, 2, \ldots, n)\) such that:

\[
\forall i: \forall s: \lambda_i p_s u_i(x_{is}) = \mu_s. \tag{4}
\]

Given concavity of the individual vNM functions, condition (4) is also sufficient. It defines a surface in expected-utility space \((v_1, \ldots, v_m)\), which we shall call the efficiency frontier of the given prospect, and denote \(\theta(L)\). Efficiency involves optimal risk-sharing among group members. Roughly, those who are relatively less risk-averse will, under an efficient contingent distribution, bear relatively more of the risk.

The efficiency frontiers of different prospects may intersect, as illustrated for the 2-person case in Fig. 1. Here, efficiency implies that the agreed choice of prospect \(L\) must be associated with contingent distributions relatively favourable to individual 1, while the choice of prospect \(K\) must be associated with distributions favourable to individual 2. An important property of CARA utility functions is that efficiency frontiers never intersect in this way, so that efficiency suffices to determine the agreed prospect, regardless of its agreed contingent distribution.

Consider an alternative prospect:

\[
K = [Y_1; \ldots; Y_m; q_1; \ldots; q_n]. \tag{5}
\]

For the group’s choices to be EU-consistent means that there exists some vNM function \(\gamma(.)\) defined over prizes, such that for any pair of prospects \((L, K)\) the group prefers \(L\) to \(K\) if:

\(^1\)Strictly, this should be “if and only if”, but EU-consistency in this stronger sense cannot be established from efficiency alone. However, except to assert that the required further assumption is fairly weak, we shall ignore this issue for the sake of conciseness.
To say that the group "prefers \textbf{L} to \textbf{K}" means that \textbf{K} will not be its agreed choice if \textbf{L} is available.

3. CARA and EU-consistency

If individual \(i\)'s preferences exhibit CARA, then they are representable by the vNM utility function:

\[
u_i(x_{is}) = \frac{1}{r} e^{-r x_{is}}.
\]

where \(r > 0\) is the Arrow–Pratt measure of absolute risk aversion. Given this, the efficiency condition (4) implies that for every pair \(\{i, j\}\) of group members:

\[
\exists \lambda_i, \lambda_j; \forall s: \lambda_i r e^{-r x_{is}} = \lambda_j r e^{-r x_{js}},
\]

which may be rewritten in logarithmic form:

\[
\exists \lambda_i, \lambda_j; \forall s: \frac{r_i}{r_j} x_{is} + \frac{\ln(\lambda_j r_j) - \ln(\lambda_i r_i)}{r_j}.
\]

Summing (9) over all \(j\) (including \(i\)):

\[
\forall s: X_s = \frac{r_i}{R} x_{is} + \sum_j \left[ \frac{\ln(\lambda_j r_j) - \ln(\lambda_i r_i)}{r_j} \right].
\]
This in turn implies that:

\[ \forall i: \forall s: x_{is} = k_i + \frac{R}{r_i} X_s, \]

where:

\[ k_i = \frac{R}{r_i} \sum_j \left[ \ln(\lambda r_i) - \frac{\ln (\lambda r_j)}{r_j} \right]. \] (13)

Thus, efficient risk-sharing entails that each individual \( i \) is allocated a fixed, state-independent payment \( k_i \), plus a uniform proportion \( R/r_i \) of each contingent prize. Across all individuals, the proportions sum to unity, while the fixed payments sum to zero. The proportions depend on, and only on, the given risk-aversion parameters \( (r_1, \ldots, r_m) \). So various efficient contingent distributions of the prospect must differ only in the fixed payments \( (k_1, \ldots, k_m) \) and, furthermore, to any zero-sum vector \( (k_1, \ldots, k_m) \) there corresponds an efficient contingent distribution.

From (12) it follows that:

\[ u_i(x_{is}) = -e^{-r_i k_i e^{-RX_i}}, \] (14)

and thus that:

\[ v_i = e^{-r_i k_i} \sum p_s \gamma(X_s), \] (15)

where:

\[ \gamma(X_s) = -e^{-RX_s}. \] (16)

The value of \( \gamma(X_s) \) depends only on the value of the prize and the given risk-aversion parameters \( (r_1, \ldots, r_m) \). Now consider any pair of prospects \( \{L, K\} \) such that:

\[ \sum_{s=1}^{n} p_s \gamma(X_s) > \sum_{s=1}^{n'} q_s \gamma(Y_s). \] (17)

To a given vector of fixed payments \( (k_1^0, \ldots, k_m^0) \) there corresponds an efficient contingent distribution of each of \( L \) and \( K \). Denote these \( [x_{is}^0] \) and \( [y_{is}^0] \) respectively. But, given (17), it follows from (15) that all individuals in the group prefer \( [x_{is}^0] \) to \( [y_{is}^0] \). Since this is the case for any given

\(^2\)Note that \( R \), which turns out to be the group’s collective degree of absolute risk-aversion, is \( 1/m \) times the harmonic mean of the individual risk-aversion measures \( r_i \).
vector of fixed payments, it follows that \( \theta(L) \) lies everywhere above \( \theta(K) \), and that efficiency alone rules out \( K \) as an agreed choice from any set of available prospects which includes \( L \).

Thus the group’s agreed choices must be EU-consistent, and representable by the vNM utility function (16), which itself exhibits CARA, of degree \( R \).

4. Non-CARA preferences and EU-inconsistency

Without the CARA assumption the group’s agreed choices are not generally EU-consistent. To see this, consider the following prospects:

\[
A = [32; 1] \quad B = [80, 0; 0.5, 0.5],
\]

and a pair of individuals \{1, 2\} with (non-CARA) utility functions:

\[
u_1 = (10 + x_1)^{0.2} \quad u_2 = (10 + x_2)^{0.6}.
\]

In this case, the efficiency frontiers \( \theta(A) \) and \( \theta(B) \) intersect (see footnote 4), so the assumption of efficiency does not suffice to determine the pair’s agreed choice. Analysis of EU-consistency therefore requires a more substantive theory of bargaining. The axiomatic theory of Nash may be construed as a model of collective rationality, so it is an obvious choice in this context.

According to the (symmetric) Nash theory, the pair will agree a prospect/distribution which maximises:

\[
N = (v_1 - d_i)(v_2 - d_i),
\]

where \( d_i \) is the expected utility, for individual \( i \), corresponding to the default outcome in the event that the pair fails to agree a choice. Assume that in this event each individual receives a zero payment \( x_i = 0 \), so that:

\[
d_1 = 1.585 \quad d_2 = 3.981.
\]

Table 1 shows the \( N \)-maximising (contingent) distributions of each prospect.³

The corresponding maximised values of \( N \) are:

\[
N(A) \approx 1.034 \quad N(B) \approx 1.024.
\]

Thus, according to the Nash theory, the pair prefers \( A \) to \( B \).⁴

Now consider two other prospects:

\[
C = [32, 0; 0.1, 0.9] \quad D = [80, 0; 0.05, 0.95].
\]

³The relevant computations were carried out using the MAPLE maths program. Numbers reported here are rounded to as few significant figures as are required to verify the Nash values. The full MAPLE worksheets are available on request from the author.

⁴This confirms that \( \theta(A) \) and \( \theta(B) \) intersect. Since \( N(A) > N(B) \), at its \( N \)-maximising distribution \( \theta(A) \) must lie above \( \theta(B) \). But allocating all of \( A \) to individual 2, which is efficient, is dominated by allocating (almost) all of \( B \) to individual 2, so in that region \( \theta(B) \) must lie above \( \theta(A) \).
Table 1
Nash bargaining solutions from among contingent distributions of prospects A and B

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th></th>
<th>B</th>
<th></th>
</tr>
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<tr>
<td></td>
<td>x_{11}</td>
<td>v_i</td>
<td>x_{11}</td>
<td>x_{12}</td>
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<td>1.900</td>
<td>30.238</td>
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Table 2
Nash bargaining solutions from among contingent distributions of prospects C and D

<table>
<thead>
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<th>C</th>
<th></th>
<th>D</th>
<th></th>
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<td>x_{12}</td>
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</tbody>
</table>

Table 2 shows their respective N-maximising contingent distributions. The corresponding maximised values of N are:

\[ N(C) = 0.0110 \quad N(D) \approx 0.0112. \]  \hspace{1cm} (24)

Thus, according to the Nash theory, the pair prefers D to C.

However, this pattern of choices is EU-inconsistent, because it violates the axiom of Independence, in a manner known as the common-ratio effect. C and D may be redefined as compound prospects:

\[ C = [A, [0; 1]; 0.1, 0.9] \quad D = [B, [0; 1]; 0.1, 0.9], \]  \hspace{1cm} (25)

so that Independence requires that A is preferred to B if and only if C is preferred to D.

5. An apparently contradictory result

We have shown that risk-sharing CARA individuals will agree choices that are EU-consistent, under any ex ante efficient bargaining procedure. This represents a wide class of bargaining theories, including that of Nash.

However, Harsanyi (1955) demonstrated for that collective choices to be both ex ante efficient and EU-consistent requires that they be representable by a linear utility-aggregation function such as:

\[ W = \sum_{i=1}^{m} \alpha_i v_i, \]  \hspace{1cm} (26)

The corresponding Nash function (20) is clearly not linear. Harsanyi’s theorem thus implies that Nash choices will be EU-inconsistent, regardless of the form of individuals’ utility functions.
The explanation for this apparent contradiction lies in our definition of EU-consistency, which is quite specific to the context of financial prospects. Harsanyi considers a more abstract and general problem, wherein the objects of choice are (lotteries over) discrete outcomes, each of which corresponds to a single point in expected-utility space. In the present context, however, the objects of choice are (lotteries over) distributable monetary prizes \( X_r \), each of which corresponds to a locus \( \theta([X_r; 1]) \) in expected-utility space.

EU-consistency in Harsanyi’s sense could be applied in the present context. It would be representable by a vNM function \( g(x_1, \ldots, x_m) \) which, as Harsanyi shows, would require a linear aggregation of individuals’ utility functions. We have found, however, that there is a vNM function \( g(X) \) corresponding to any efficient utility-aggregation rule or bargaining process, if all individuals’ preferences are CARA.

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References