Minimum Distribution-Sensitivity, Poverty Aversion, and Poverty Orderings

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This paper introduces a measure of distribution-sensitivity, which is similar to Arrow-Pratt’s measure of risk aversion, for a poverty index. The measure also gauges poverty aversion and has a clear and straightforward interpretation. Using this measure, we can define various classes of minimum distribution-sensitive poverty indices. We show that the ordering condition for such a class of poverty indices is simply a generalized (poverty) deprivation profile dominance. Properties and implications of this dominance condition are analyzed and poverty orderings by different classes of poverty measures are compared. Journal of Economic Literature Classification Number: I32. © 2000 Academic Press

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1. INTRODUCTION

Pigou-Dalton’s principle of transfers has played a central role in the literature of income distribution and welfare. The principle states that a progressive transfer of income between two persons reduces income inequality and enhances social welfare of the distribution. Sen [28] introduced a limited form of the principle into poverty measurement and labeled it the “transfer axiom.” This axiom requires that a transfer of income from a poor person to someone poorer reduce poverty. Sen justified this axiom by persuasively arguing that a poverty index should not only reflect the incidence of poverty but also be sensitive to income redistribution within the poor. Since then, this axiom has been regarded as an indispensable requirement in poverty measurement and a satisfactory poverty index is said to be distribution-sensitive.

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In the past two decades following Sen’s pioneering work, researchers have proposed additional axiomatic requirements and developed various classes of distribution-sensitive poverty indices.\(^2\) Recently, however, much of the effort has switched from the construction of new indices to the exploration of poverty orderings.\(^3\) This switch arises partly from the fact that Sen’s axiomatic framework does not lead to a unique poverty index; there exist multiple satisfactory poverty indices for any given set of reasonable axioms. For example, the transfer axiom does not specify how much poverty should be reduced for a given transfer between two persons as long as the transfer reduces poverty, which leaves a large room for discretion in choosing a poverty index. As a consequence, it may happen that two equally “desirable” indices indicate opposite conclusions in ranking distributions. Thus it is useful to identify situations where distributions can be unanimously ranked for all members of a given class of poverty indices.

In an important contribution, Atkinson [3] identified such situations, inter alia, for all additively separable distribution-sensitive indices with the same poverty line: the necessary and sufficient condition is a second degree stochastic dominance relation among the distributions up to the poverty line. He also pointed out that once this ordering condition holds at a given poverty line \(z\) it will also hold for any other poverty line below \(z\). Interestingly, as demonstrated nicely by Foster and Shorrocks [13, 14], second degree stochastic dominance is also necessary and sufficient for a member of the Foster et al. [10] indices to indicate a consistent poverty ranking for all common poverty lines in \((0, z]\).

In this paper, we follow Atkinson and explore poverty orderings for additively separable poverty indices with a common poverty line. Analogous to Meyer’s [22] usage of minimum risk averse utility function, we consider poverty indices possessing a pre-determined minimum distribution-sensitivity. This consideration ensures the transfer axiom to be properly satisfied by all poverty indices. The measure of distribution-sensitivity is similar to the Arrow-Pratt measure of risk aversion in the literature of risk and uncertainty. It quantifies the sensitivity of a poverty index to income transfers: the larger the measure the more sensitive the poverty index is to income transfers. A poverty index is more distribution-sensitive than another if and only if the former is a strictly convex transformation of the latter. The measure of distribution-sensitivity can also be interpreted as a measure of “poverty aversion,” a concept which has been frequently used but a formal definition is lacking. Thus a poverty index

\(^2\) For a survey, see Chakravarty [5], Foster and Sen [12], or Zheng [39].
\(^3\) Contributions to this topic include Atkinson [3], Foster and Shorrocks [13, 14], Jenkins and Lambert [16, 17], Shorrocks [30, 31], Foster and Jin [11], Howes [15], Mitra and Ok [23] and Zheng [40, 42]. A review of this literature can be found in Zheng [41].
is more “poverty-averse” than another if and only if the former is more distribution-sensitive than the latter.

This paper is also motivated in part by a result in Zheng [42] on increasing the ordering power of poverty beyond the Atkinson condition by resorting to higher degrees of stochastic dominance. It is also closely related to a recent interesting work by Foster and Jin [11] on poverty orderings for the Dalton utility-gap poverty indices.

The rest of the paper is organized as follows. Section 2 defines poverty indices and other notations. It also introduces the measure of distribution-sensitivity (poverty aversion) and examines its implications for poverty indices. Section 3 derives the ordering condition for a class of minimum distribution-sensitive poverty indices. Various properties of this ordering condition are also explored. Section 4 concludes the paper with some remarks.

2. DISTRIBUTION-SENSITIVITY AND POVERTY INDICES

Consider an income distribution of \( n \) individuals \( X = (x_1, x_2, \ldots, x_n) \) where \( x_i \) is drawn from some nonnegative non-degenerate real interval \( D \subseteq \mathbb{R}_+ \). Without loss of generality, we further assume that \( X \) is in ascending order, i.e., \( x_1 \leq x_2 \leq \cdots \leq x_n \). For convenience, we restrict our attention to distributions with the same fixed dimension \( n \); it is easy to see that our results are also valid for distributions with different and variable dimensions. Thus the set of income distributions we consider is \( \Psi^n : = \{ X = (x_1, x_2, \ldots, x_n) \mid x_i \in D \} \).

A poverty index is a function \( P \) whose value indicates the degree of poverty associated with a given distribution and a poverty line. For an income distribution \( X \) and a poverty line \( z \), an additively separable poverty index is

\[
P(X; z) = \frac{1}{n} \sum_{i=1}^{n} p(x_i, z),
\]

where \( p(x, z) \) is the individual deprivation function with \( p(x, z) > 0 \) on \( [0, z] \) and \( p(x, z) = 0 \) for \( x \geq z \).

We further assume that \( p(x, z) \) is twice differentiable with respect to \( x \) for \( x < z \) and is differentiable with respect to \( z \), and the differentiations are denoted as \( p_x, p_{xx} \) and \( p_z \), respectively.

In this paper, we use the weak definition of the poor, i.e., the person at the poverty line is regarded nonpoor. This practice makes it possible to consider the censored income distribution instead of the original distribution. See Donaldson and Weymark [9] for further discussions on the definition of the poor.
The poverty index \( P \) defined in (2.1) is clearly symmetrical and satisfies the focus axiom which requires \( P \) to be independent of the income distribution above the poverty line. \( P \) also satisfies the replication invariance axiom which requires that the pooling of several identical populations leave poverty unchanged. If \( p(x, z) \) is continuous as a function of \( x \in [0, \infty) \) then \( P \) satisfies the continuity axiom; if further \( p_x < 0 \) for all \( x \in [0, z) \) then \( P \) satisfies the monotonicity axiom; if additionally \( p_{xx} > 0 \) for all \( x \in [0, z) \) then \( P \) satisfies the transfer axiom.\(^5\) Finally if \( p_z > 0 \) for all \( z \in [0, \infty) \) then \( P \) satisfies the increasing poverty line axiom. We assume that \( P \) possesses all these properties.

2.1. The Measure of Distribution-Sensitivity

Given two poverty indices that satisfy the transfer axiom and other poverty axioms, how do we characterize their difference? Why could these two indices rank distributions differently? In this section we argue that, under a certain assumption, this difference is reflected in their sensitivity to income transfers. That is, if the \( p \) function is unique up to a linear transformation then a poverty index can be completely characterized by the way it reacts to income transfers. The degree of this reaction is referred to in this paper as the distribution-sensitivity of the poverty index.

To understand the notion of distribution-sensitivity, it is helpful to state the transfer axiom in a different yet equivalent form: for two poor persons with incomes \( x_1 < x_2 < z \), a simple increment of income to \( x_1 \) reduces poverty more than the same-amount increment to \( x_2 \).\(^6\) This is equivalent to requiring \( p_x(x_1, z) < p_x(x_2, z) < 0 \) for \( x_1 < x_2 \). Thus the distribution-sensitivity of a poverty index means that the poverty reduction for a simple increment of income is sensitive not only to the amount of the increment but also to the income level of the poor. It follows immediately from this interpretation that the measure of distribution-sensitivity would be some kind of relative difference in poverty reduction between \( x_1 \) and \( x_2 \) for a small increase of income \( dx \) to each person.\(^7\) A moment’s consideration suggests that the only sensible measures would be, naturally, \( \frac{p_x(x_1, z) - p_x(x_2, z)}{p_x(x_2, z)} \) \( dx \) relative to \( p_x(x_2, z) \) \( dx \) or \( p_x(x_1, z) \) \( dx \) or an

\(^5\) Note that if continuity is replaced with restricted continuity, i.e., \( p(x, z) \) is continuous as a function of \( x \) only for all \( x \in [0, z) \), then \( P \) satisfies the weak monotonicity axiom and the weak transfer axiom. In both cases, no one changes poverty status as a result of the income change. In this paper we maintain continuity and hence do not consider axioms in their weak form.

\(^6\) This is the so-called “monotonicity sensitivity axiom” as proposed by Kakwani [18]. The equivalence of this axiom to the transfer axiom is stated in Kakwani [18] and is also discussed in Zheng [39].

\(^7\) The consideration of the absolute difference would lead to \( p_{xx} \), which can be ruled out as a measure of distribution-sensitivity since \( p_{xx} \) is not unique up to a linear transformation.
average of both. These measures represent the (relative) marginal reduction in poverty if $dx$ is added to $x_1$ as opposed to the case if $dx$ is given to $x_2$. It follows that between two poverty indices, the one with a larger measure will indicate greater reduction in poverty if the income increment is given to a poorer person. Thus, measures of relative difference indeed capture the notion of distribution-sensitivity as we have interpreted above. Since the base to which the absolute difference is compared is not important, we may just consider the base $p_x(x_1, z)$ for convenience.

Furthermore, if we let $x_2 = x_1 + dx$, then we have

$$[p_x(x_1, z) - p_x(x_2, z)] dx / [p_x(x_1, z) dx] \approx [p_x(x_1, z) / p_x(x_1, z)] dx.$$  

Thus we justify $-p_x(x_1, z) / p_x(x_1, z)$ as a measure of distribution-sensitivity at income $x_1$. Formally, we may define the measure of distribution-sensitivity for a poverty index $p(x, z)$, for all $x < z$, as follows:

$$s_p(x, z) = -\frac{p_x(x, z)}{p_x(x, z)}.$$  

The measure of distribution-sensitivity $s_p(x, z)$ is very much similar to the Arrow-Pratt measure of absolute risk aversion $r(x) = -u''(x)/u'(x)$ for a utility function $u(x)$ with $x$ being wealth. In fact it is obtained from $r(x)$ by replacing $u(x)$ with the individual poverty deprivation function $p(x, z)$. Consequently, an important property of $r(x)$ is carried over to $s(x, z)$: if $u(x)$ is more risk averse than $v(x)$ everywhere then $u(x)$ must be a concave function of $v(x)$.

**Proposition 1.** For a given poverty line $z$ and two poverty deprivation functions $p(x, z)$ and $q(x, z)$, the following two statements are equivalent:

(i) $s_p(x, z) \geq s_q(x, z)$ for all $x \in [0, z]$.

(ii) $p(x, z)$ is a [strictly] convex function of $q(x, z)$ for $x \in [0, z]$.

The proof is the same as in Pratt [25]. Note that in (ii) the function $p$ is “convex” rather than “concave” as in Pratt’s condition because $p(x, z)$ is a decreasing function of $x$.

The measure of distribution-sensitivity defined in (2.2) can also be interpreted as a measure of “poverty aversion.” The term “poverty aversion” has been frequently employed in the literature to explain the parameters embedded in poverty indices (e.g., Seidl [27] and Dagum [7]). Seidl also proposed an axiom of “increasing poverty aversion.” Despite these, a careful review of the literature suggests that the concept of poverty aversion has not been formally defined. Although “poverty aversion” seems to have been

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8 In the limiting case as $x_2 \to x_1$, it makes no difference whether the base is $p_x(x_2, z) dx$ or $p_x(x_1, z) dx$ or an average of both.
coined from “risk aversion,” an interpretation from the perspective of an individual poor person, similar to that of “risk aversion,” does not exist. It is reasonable, however, to interpret the concept from the perspective of a social planner whose objective is to enhance the social welfare of the poor.9

A poverty-averse social planner foremost dislikes poverty, i.e., he desires all poor incomes to be raised to the poverty line. Such an interpretation amounts to assuming the monotonicity axiom. However, the notion of “poverty aversion” that researchers have in mind means a lot more than just “disliking poverty.” It seems that the following additional interpretation is needed to capture the notion: if $S$ has to be collected from one of two poor persons, then a caring social planner is less likely to get it from the poorer one. Or in other words, the social planner feels that the social welfare of the poor as a group becomes worse-off if $S$ is taken away from the poorer person than from the less poor person. Clearly such an interpretation of “poverty aversion” coincides with the description of distribution-sensitivity. Thus, “poverty aversion” and distribution-sensitivity may be regarded as the two sides of the same coin and can be used interchangeably. It also follows directly from Proposition 1 that a poverty index is more poverty-averse than another index if and only if the former is a convex transformation of the latter.10

2.2. Distribution-Sensitivity and Poverty Indices

Using the measure of distribution-sensitivity $s(x, z)$, we can characterize various distribution-sensitive poverty indices. This characterization is unique up to a linear transformation in the sense that two poverty indices having the same distribution-sensitivity if and only if they are related by a linear transformation. The characterization also allows us to group poverty indices according to some pre-specified requirement on distribution-sensitivity or poverty aversion.

The transfer axiom only requires that a progressive (regressive) transfer of income decrease (increase) poverty; it does not specify the extent to

9 This interpretation, however, departs from the usual characterization of poverty (e.g., Sen [28]) as the relative deprivation of the poor for living below the poverty line.

10 Seidl [27] defined “more poverty aversion” of $p(x, z)$ than $q(x, z)$ as, in our terms, that the ratio $\frac{p(x, z)}{q(x, z)}$ increases as $x$ decreases for a given positive $\delta$. Since $p$ and $q$ are continuous functions of $x$ for $x \in [0, z]$, it follows that $\frac{p(x, z)}{q(x, z)}$ must be a decreasing function of $x$, which implies $-\frac{p(x, z)}{q(x, z)} > -\frac{q(x, z)}{p(x, z)}$. Hence, his definition does not capture the notion of poverty aversion as we discuss here; it only says that the social planner with $p$ dislikes poverty more than the social planner with $q$. However, if “more poverty aversion” is defined in terms of $\frac{p(x, z) - p(x, z)}{q(x, z) - q(x, z)}$ for any positive $\delta$, then “more poverty aversion” implies that $\frac{p(x, z)}{q(x, z)}$ is an increasing function of $x \in [0, z]$. This will necessarily lead to condition (i) of Proposition 1 and, hence, the definition becomes equivalent to ours and provides an alternative interpretation of “poverty aversion.”
which the decrease or increase should be. Thus the axiom amounts to requiring \( s(x, z) > 0 \) for a poverty index. Since the monotonic poverty indices corresponding to \( s(x, z) = 0 \) are positive linear functions of \( p(x, z) = z - x \) (the poverty gap), the transfer axiom can also be interpreted as to requiring a poverty index to be more distribution-sensitive than \( p(x, z) = z - x \). Since the poverty gap is not distribution-sensitive, the requirement \( s(x, z) > 0 \) is not very demanding. In fact all commonly used distribution-sensitive poverty indices are bounded well above the index of poverty gap in distribution-sensitivity. In what follows we examine these indices.

The Clark et al. [6] index corresponds to \( p(x, z) = \frac{1}{2} [1 - (x/z)^{\beta}] \) with \( \beta < 1 \), its measure of distribution-sensitivity is \( s_d(x, z) = (1 - \beta)/x \). Clearly, the poverty index becomes more distribution-sensitive as \( \beta \) gets smaller. In this sense, \( (1 - \beta) \) is usually interpreted as the parameter of poverty aversion: for a given \( x \), the smaller is \( \beta \) the more poverty-averse is the index. Since the Watts [38] index corresponds to \( \beta = 0 \), the Clark et al. index with a negative \( \beta \) is more distribution-sensitive than the Watts index, which in turn is more distribution-sensitive than the Clark et al. index with a positive \( \beta \), or the Chakravarty [4] index. Conformed with Proposition 1, the Clark et al. index with a negative \( \beta \) is an increasing and convex function of the Watts index, which is an increasing and convex function of the Chakravarty index.

For the Foster et al. [10] class of poverty indices, \( p(x, z) = (1 - x/z)^{\alpha} \) with \( \alpha > 1 \), the measure of distribution-sensitivity is \( s_f(x, z) = (\alpha - 1)/(z - x) \). Thus the index becomes more distribution-sensitive with increasing \( \alpha \) and the value of \( (\alpha - 1) \) can be taken as an indicator of poverty aversion; if \( \alpha \) were to take value 1 then the Foster et al. index becomes the poverty gap ratio and exhibits no poverty aversion. It is also useful to compare the sensitivity of the Foster et al. index with the Clark et al. class. For given values of \( \alpha \), \( \beta \) and \( z \), neither index is more distribution-sensitive than the other for all \( x \in [0, z) \). This is because \( s_f(x, z) > s_d(x, z) \) for \( x > \frac{1 - \beta}{\alpha - \beta} z \) and \( s_f(x, z) < s_d(x, z) \) for \( x < \frac{1 - \beta}{\alpha - \beta} z \). Thus, between these two indices, the Clark et al. index is more distribution-sensitive at low income levels while the Foster et al. index is more distribution-sensitive at high income levels. Interestingly, as \( \alpha \to \infty \) with fixed \( \beta \) and \( z \to z \), the Foster et al. index becomes more distribution-sensitive for all \( x \in [0, z) \). On the other hand, as \( \beta \to -\infty \) with fixed \( \alpha \) and \( z \to z \), the Clark et al. index becomes more distribution-sensitive for all \( x \in [0, z) \). These observations are not surprising because it can be shown (see, e.g., Zheng [39, pp. 151–152]) that

\[ s_f(x, z) = \frac{\alpha - 1}{\alpha - \beta} \] for \( x < z \) since \( p(x, z) = 0 \) for \( x \geq z \) by the focus axiom.
both the Foster et al. index with \( x \to \infty \) and the Clark et al. index with \( \beta \to -\infty \) approach the Rawlsian maximin justice. It is also useful to note that as \( x \) increases, the Clark et al. index exhibits diminishing poverty aversion while the Foster et al. index indicates increasing poverty aversion.\(^{12}\)

The poverty indices that are of particular interest in this paper are the CDS (constant distribution-sensitivity) class of poverty indices \( p(x, z) = e^{\gamma(x-z)} - 1 \) with \( \gamma > 0 \). This class has not received much attention in the literature and was only recently characterized by Zheng [41] as coherent with infinite degree stochastic dominance. These indices are special because the measure of distribution-sensitivity is constant, that is \( s(x, z) = \gamma \). Thus, the marginal reduction in poverty if income is added to \( x_1 \) instead of \( x_2 \) (\( x_1 < x_2 \)) will always be the same regardless of \( x_1 \) and \( x_2 \) as long as \( x_2 - x_1 \) remains constant. It is easy to verify that at a lower income level the CDS poverty index is more poverty-averse than the Foster et al. index but is less poverty-averse than the Clark et al. index.

All of the above poverty indices, except the Foster et al. class with \( x > 1 \), also belong to the general Daltonian class of poverty indices. A poverty index \( P \) is a Dalton-type if, for \( x < z \), the individual deprivation function can be expressed as \( p(x, z) = A(z)\left[ \hat{p}(z) - \hat{p}(x) \right] \) for some functions \( \hat{p}(\cdot) \) and \( A(\cdot) \). Clearly, in order for such a poverty index to satisfy continuity, monotonicity and strong transfer, \( \hat{p}(x) \) must be twice differentiable in \( x \) with \( \hat{p}_x > 0 \) and \( \hat{p}_{xx} < 0 \). The normalization factor \( A(z) \) also must be properly specified to ensure that \( P \) satisfies the axiom of increasing poverty line.\(^{13}\)

The measure of distribution-sensitivity for a Daltonian poverty index is simply \( s(x, z) = -\hat{p}_{xx}/\hat{p}_x \) for \( x < z \).

3. POVERTY ORDERINGS FOR THE MINIMUM DISTRIBUTION-SENSITIVE INDICES

In the literature, there are three levels of poverty orderings for a class of poverty indices: weak poverty orderings (e.g., Atkinson [3]), semi-strict poverty orderings (e.g., Howes [15]) and strict poverty orderings (e.g.,

\(^{12}\) This is to say that the (relative) marginal reduction in poverty as measured by \( s(x, z) \) decreases for the Clark et al. index while increases for the Foster et al. index as \( x \) increases.

\(^{13}\) The commonly used normalization factor \( A(z) \) is either 1 or \( 1/\hat{p}(z) \). For poverty indices with these two \( A(z) \)'s, the axiom of increasing poverty line is clearly satisfied.
Zheng [42]). For two income distributions $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ and a fixed poverty line $z$, weak orderings require $P(X; z) \geq P(Y; z)$ for all poverty indices $P$ in a given class; semi-strict orderings require, in addition, that the strict inequality hold for some poverty indices; strict orderings require that the strict inequality hold for all poverty indices. In this paper, for simplicity of presentation, we adopt the weak definition of poverty orderings.

The poverty ordering conditions for all distribution-sensitive poverty indices have been well established in the literature. For a common fixed poverty line $z$, Atkinson [3] showed that the necessary and sufficient condition for $P(X; z) \geq P(Y; z)$ for all poverty indices with $s(x, z) \geq 0$ is $\int_0^z F_X(t) \, dt \geq \int_0^z F_Y(t) \, dt$ for all $z \in [0, z]$ where $F_X$ and $F_Y$ are the cumulative distribution functions of $X$ and $Y$, respectively. Foster and Shorrocks [14] noted that this stochastic dominance condition is equivalent to censored generalized Lorenz dominance. That is, the necessary and sufficient condition for $P(X; z) \geq P(Y; z)$ for all poverty indices with $s(x, z) \geq 0$ is $\sum_{i=1}^k (z - \hat{x}_i) \geq \sum_{i=1}^k (z - \hat{y}_i)$ for $k = 1, 2, \ldots, n$, where $\hat{x}_i$ and $\hat{y}_i$ are the censored incomes of $x_i$ and $y_i$ at $z$. Recently, Spencer and Fisher [33], Jenkins and Lambert [16], and Shorrocks [31] characterized a new dominance curve, (poverty) deprivation profile as termed by Shorrocks, which is based on the cumulative poverty gap, $\frac{1}{n} \sum_{i=1}^n (z - \hat{x}_i)$. They showed that this condition is equivalent to the censored generalized Lorenz dominance condition. Consequently, the results most resemble the conditions that we will derive, for ease of reference, we first summarize their results as follows.

**Proposition 2** (Spencer and Fisher [33], Jenkins and Lambert [16], and Shorrocks [31]). The necessary and sufficient condition for $P(X; z) \geq P(Y; z)$ for all poverty indices with $s(x, z) \geq 0$ and a given $z$ is that $\sum_{i=1}^k (z - \hat{x}_i) \geq \sum_{i=1}^k (z - \hat{y}_i)$ for $k = 1, 2, \ldots, n$.

Clearly, the poverty deprivation dominance condition is equivalent to censored generalized Lorenz dominance. Interestingly, as shown in Foster and Shorrocks [13, 14], the censored generalized Lorenz dominance condition is also necessary and sufficient to entail $P(X; z_1) \geq P(Y; z_1)$ for all $z_1 \in (0, z)$ with $P(x, z_1) = 1 - x/z$ for $x < z$. Also an immediate and useful corollary of Proposition 2, as noted by Atkinson [3] and others, is

**Corollary 2.1.** $P(X; z_1) \geq P(Y; z_1)$ for all poverty indices with $s(x, z) \geq 0$ implies, for any $z_2 \in (0, z_1]$, $P(X; z_2) \geq P(Y; z_2)$ for all poverty indices with $s(x, z) \geq 0$.

As shown in Zheng [42], the results established for weak poverty orderings may not hold for the other two definitions of poverty orderings. While we do not pursue it here, results for the other two definitions of poverty orderings can be similarly derived.
3.1. The Class of Minimum Distribution-Sensitive Poverty Indices

While Atkinson considered poverty indices with \( s(x, z) \geq 0 \), we may consider poverty indices with \( s(x, z) \geq l(x, z) \) and \( l(x, z) \) is nonzero. For example, \( l(x, z) \) may be selected as \((1 - \beta)/x, (x - 1)/(z - x)\) or simply a constant \( \gamma \). We refer to a poverty index satisfying \( s(x, z) \geq l(x, z) \) as a poverty index with minimum distribution-sensitivity \( l(x, z) \). Since the function \( l(x, z) \) uniquely (up to a linear transformation) determines a poverty index, say, \( q(x, z) \), all poverty indices with minimum distribution-sensitivity \( l(x, z) \) are necessarily increasing and convex functions of \( q(x, z) \).

We denote \( \mathcal{E}(q(x, z)) \) (or simply \( \mathcal{E}(q) \)) as the class of all poverty indices with \( q(x, z) \) as the lower bound of distribution-sensitivity. Formally,

\[
\mathcal{E}(q(x, z)) := \{ P(X; z) = \frac{1}{n} \sum_{i=1}^{n} p(x_i, z) \mid s_q(x, z) \geq s_q(x, z) \text{ for all } x \in [0, z) \}.
\] (3.1)

Clearly, each \( p(x, z) \) is also continuous in \( q(x, z) \) because \( p \) is a convex transformation of \( q \). If \( q(x, z) = z - x \), then \( \mathcal{E}(q) \) and the class of distribution-sensitive poverty indices considered by Atkinson [3] coincide.

One reason for specifying a non-zero minimum distribution-sensitivity requirement for a poverty index is that one may regard the usual requirement \( s(x, z) \geq 0 \) as too loose. This is particularly true when the weak definition of the transfer axiom is considered as in Atkinson [3] and Howes [15]. In this case, the poverty gap ratio itself is included in \( \mathcal{E}(q) \). Even when the strong definition of the transfer axiom (the one used in this paper) is adopted, it is still desirable to be able to exclude those poverty indices which are “too close” to the poverty gap ratio.

Another reason for considering the class \( \mathcal{E}(q) \) is that poverty orderings for minimum distribution-sensitive indices may substantially enhance the power or the completeness of poverty orderings beyond the Atkinson second degree condition. To increase the power, one needs of course to

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15 In practice, the determination of \( q(x, z) \) could be quite difficult. One may, however, follow Atkinson [2, p. 67] and use the leaky-bucket experiment to obtain the parameter in \( q(x, z) \) once the functional form of \( q(x, z) \) is known. The idea of the experiment is that if a poverty index is distribution-sensitive, then the poverty level may remain unchanged if \( y \) is taken away from someone with income \( y \) but only a portion of \( y \), say \( \rho \in [0, 1) \), is given to someone with income \( x \). The more distribution-sensitive is \( q(x, z) \), the greater is the difference \( 1 - \rho \) for given levels of \( x \) and \( y \). Different social planners may have different values of \( 1 - \rho \) and, thus, the smallest \( 1 - \rho \) will determine the minimum distribution-sensitive poverty index. Amiel et al. [1] have used the leaky-bucket experiment to determine parameters in the Atkinson inequality indices and the Gini-type social welfare function.

16 The weak definition of the transfer axiom requires \( p_{\rho}(x, z) \geq 0 \) for all \( x \in [0, z) \). The strong definition is the one used in this paper and requires \( p_{\rho}(x, z) > 0 \) for all \( x \in [0, z) \).
limit the admissible poverty indices. This can be achieved by introducing some more restrictive poverty axioms than the transfer axiom and, consequently, using higher degree stochastic dominance conditions (e.g., Ravallion [26]). However, third and higher degrees of stochastic dominance all include a second-degree condition at the poverty line. Thus if the poverty line is uncertain and is allowed to vary between 0 and the maximum poverty line, higher degrees of stochastic dominance will collapse to the second degree. Clearly, a solution to this problem is to remove the second-order condition from third or higher degree stochastic dominance, but this requires some strong poverty axioms which may not be unconditionally justified (Zheng [42]). Alternatively, one may limit poverty indices to those satisfying some minimum distribution-sensitivity requirement.

3.2. Poverty Orderings for the Class $\mathcal{Z}(q)$

The following proposition characterizes poverty orderings for all poverty indices in the class $\mathcal{Z}(q(x, z))$.

**Proposition 3.** For two distributions $X \in \mathcal{Y}^n$ and $Y \in \mathcal{Y}^n$ and a given poverty line $z$, the necessary and sufficient condition for $P(X; z) \geq P(Y; z)$ for all $P$ in $\mathcal{Z}(q)$ is

$$
\sum_{i=1}^{k} q(x_i, z) \geq \sum_{i=1}^{k} q(y_i, z), \quad k = 1, 2, ..., n. \tag{3.2}
$$

**Proof.** Since for any poverty index $P$ in $\mathcal{Z}(q(x, z))$, the individual poverty deprivation function $p(x, z)$ is a continuous, increasing and convex function of $q(x, z)$, the condition $P(X; z) \geq P(Y; z)$ for all $P$ in $\mathcal{Z}(q(x, z))$ can be rewritten as requiring

$$
\frac{1}{n} \sum_{i=1}^{n} \pi(q(x_i, z)) \geq \frac{1}{n} \sum_{i=1}^{n} \pi(q(y_i, z)) \tag{3.3}
$$

for all continuous, increasing and convex functions $\pi(\cdot)$. By definition, both $\{q(x_i, z)\}$ and $\{q(y_i, z)\}$ are nonincreasing in $i$. Using a standard result from the theory of majorization (Marshall and Olkin [20, Proposition B.2, p. 109]), we know that the necessary and sufficient condition for (3.3) to hold is that $\{q(y_i, z)\}$ is weakly submajorized by $\{q(x_i, z)\}$, which is exactly what condition (3.2) states.\textsuperscript{17}

\textsuperscript{17} Note that the Tomić theorem that we are using requires (3.3) to hold for all convex functions $\pi(\cdot)$. In our case, however, $\pi(\cdot)$ must satisfy $\pi(0) = 0$ by the definition of the poverty index. This does not pose a problem in using the Tomić theorem because in the proof of the theorem the convex function $g(z)$ constructed also satisfies $g(0) = 0$. 

Clearly, condition (3.2) can be regarded as a generalization of the poverty deprivation dominance condition derived by Spencer and Fisher, Jenkins and Lambert and Shorrocks; the latter condition is a special case of (3.2) with \( q(x, z) = z - x \). Thus, a dominance curve similar to the deprivation profile can be constructed: it is the linear segment of the coordinates \((0, 0)\) and \( \left( \frac{k}{n}, \frac{k}{n} \sum_{i=1}^{k} q(x_i, z) \right) \) for \( k = 1, 2, \ldots, n \). Thus condition (3.2) is equivalent to requiring that the generalized deprivation profile of \( X \) lie nowhere below that of \( Y \).

Unlike the poverty deprivation dominance condition which is equivalent to (censored) generalized Lorenz dominance, condition (3.2) generally does not have such a corresponding equivalent condition. However, if \( q(x, z) \) is a Dalton-type poverty index then such an equivalence does exist. The following corollary presents this equivalence for this special case.

**Corollary 3.1.** If \( q(x, z) = A(z)[\tilde{q}(z) - \tilde{q}(x)] \) with \( A(z) > 0 \) for \( x < z \) and \( q(x, z) = 0 \) for \( x \geq z \), then condition (3.2) and the following condition are equivalent:

\[
\sum_{i=1}^{k} \tilde{q}(\hat{x}_i) \leq \sum_{i=1}^{k} \tilde{q}(\hat{y}_i), \quad k = 1, 2, \ldots, n. \tag{3.4}
\]

This condition is clearly a generalization of the generalized Lorenz dominance condition; the latter is a special case of (3.4) with \( \tilde{q}(x) = x \). Condition (3.4) is implied by the generalized Lorenz dominance condition since \( \tilde{q} \) is continuous, increasing and concave for \( x \in [0, z] \) (Marshall and Olkin [20, Theorem A.2] and Foster and Jin [11, Theorem 2]).

For a given \( q(x, z) \) and a reference distribution \( X \), condition (3.2) can separate all other distributions in \( \mathcal{P}^n \) into two groups: those satisfying condition (3.2) and those not. If we measure the power of poverty orderings

\[18\] Naturally, the area underneath this generalized poverty deprivation curve can be used as a poverty index. It is easy to calculate that twice the area is \( 2 \sum_{i=1}^{n} (n + 0.5 - i) q(x_i, z) \). This index is not additively separable but satisfies all other poverty axioms discussed in this paper. When \( q(x, z) = z - x \), the index is what Shorrocks [30] characterized as “the modified Sen index” and is also a member of the classes that Thon [35] and Chakravarty [4] introduced.

\[19\] Condition (3.4) can be regarded as a transformed generalized Lorenz dominance or as generalized Lorenz dominance of a utility function (Foster and Jin [11]) if \( \tilde{q}(x) \) is interpreted as a utility function. The characterization of the transformed generalized Lorenz dominance is a bit different from that of the ordinary generalized Lorenz dominance; the latter uses income increments and mean-preserving transfers while the former uses utility increments and utility-preserving transfers. For example, the transformed generalized Lorenz dominance with \( \tilde{q}(x) = \ln x \) is characterized by income (utility) increments and geometric-mean-preserving transfers. Since an arithmetic-mean-preserving progressive transfer can be regarded as an income increase and a geometric-mean-preserving progressive transfer, it follows that the ordinary generalized Lorenz dominance necessarily implies the transformed generalized Lorenz dominance.
by the set of distributions in $\mathcal{P}^n$ that satisfy condition (3.2), we can compare the power of poverty orderings with different $q$ functions. An interesting question to explore is: under what condition will the set of distributions in $\mathcal{P}^n$ ordered by $\mathcal{E}(q_1(x, z))$ be included in the set ordered by $\mathcal{E}(q_2(x, z))$? The following proposition provides the answer to this question.

**Proposition 4.** For any two distributions $X \in \mathcal{P}^n$ and $Y \in \mathcal{P}^n$, given $P(X; z) \succeq P(Y; z)$ for all $P$ in $\mathcal{E}(q_1(x, z))$, the necessary and sufficient condition for $P(X; z) \succeq P(Y; z)$ for all $P$ in $\mathcal{E}(q_2(x, z))$ is that $q_2(x, z)$ is a convex function of $q_1(x, z)$.

**Proof.** Use (3.2), the above condition can be rephrased as: under what condition will $\sum_{i=1}^k q_1(x_i, z) \geq \sum_{i=1}^k q_1(y_i, z)$, $k = 1, 2, \ldots, n$, always imply $\sum_{i=1}^k q_2(x_i, z) \geq \sum_{i=1}^k q_2(y_i, z)$ for $k = 1, 2, \ldots, n$. If we regard $q_2$ as a transformation of $q_1$, then this transformation should preserve poverty orderings.

If $q_2(x, z)$ is an increasing and convex transformation of $q_1(x, z)$, then the application of Theorem A.2 of Marshall and Olkin [20, p. 116] entails that $\sum_{i=1}^k q_1(x_i, z) \geq \sum_{i=1}^k q_1(y_i, z)$, $k = 1, 2, \ldots, n$, will necessarily imply $\sum_{i=1}^k q_2(x_i, z) \geq \sum_{i=1}^k q_2(y_i, z)$, $k = 1, 2, \ldots, n$.

On the other hand, if for any distributions $X$ and $Y$ in $\mathcal{P}^n$, $\sum_{i=1}^k q_1(x_i, z) \geq \sum_{i=1}^k q_1(y_i, z)$ for $k = 1, 2, \ldots, n$ always implies $\sum_{i=1}^k q_2(x_i, z) \geq \sum_{i=1}^k q_2(y_i, z)$ for $k = 1, 2, \ldots, n$, then $q_2(x, z)$ must be an increasing and convex transformation of $q_1(x, z)$. This result follows from Moyes [24] who showed that increasing and concave transformations are necessary to preserve generalized Lorenz dominance. Since a generalized poverty deprivation profile dominance between $\{q_1(x, z)\}$ and $\{q_1(y, z)\}$ is equivalent to a generalized Lorenz dominance between $\{-q_1(x, z)\}$ and $\{-q_1(y, z)\}$, the application of Theorem 3.1 of Moyes [24] implies that $-q_2(x, z)$ is an increasing and concave transformation of $-q_1(x, z)$. It follows immediately that $q_2(x, z)$ must be an increasing and convex transformation of $q_1(x, z)$.

The result of the above proposition is very useful in comparing the power of poverty orderings with different sets of minimum distribution-sensitivity poverty indices. A direct implication of the proposition is that if $q_2(x, z)$ is an increasing and convex transformation of $q_1(x, z)$, then the poverty ordering by $\mathcal{E}(q_2(x, z))$ will have at least as much power as that by $\mathcal{E}(q_1(x, z))$. If the transformation is strictly convex then the power will be strictly increased as demonstrated in the following corollary.

**Corollary 4.1.** If $q_2(x, z)$ is a strictly convex transformation of $q_1(x, z)$, then the poverty ordering by $\mathcal{E}(q_2(x, z))$ will have strictly more power than that by $\mathcal{E}(q_1(x, z))$. 
Proof. Suppose the result is not true, that is, both $\mathcal{E}(q_2(x,z))$ and $\mathcal{E}(q_1(x,z))$ order the same set of distributions. It follows that

\[ \sum_{i=1}^{k} q(x_i, z) \geq \sum_{i=1}^{k} q(y_i, z), \quad k = 1, 2, ..., n, \]

will also always imply

\[ \sum_{i=1}^{k} q_1(x_i, z) \geq \sum_{i=1}^{k} q_1(y_i, z), \quad k = 1, 2, ..., n, \]

for any distributions $X$ and $Y$. Since $q_1(x, z)$ is also a continuous and increasing transformation of $q_2(x, z)$, then the application of the results of Moyes [24] again entails that $q_1(x, z)$ is a convex transformation of $q_2(x, z)$. Clearly it is not possible for $q_2(x, z)$ to be a strictly convex function of $q_1(x, z)$ and, at the same time, for $q_1(x, z)$ to be a convex function of $q_2(x, z)$. Thus the corollary is correct.

If there is no convexity relation between $q_1(x, z)$ and $q_2(x, z)$, then the relation between the set of distributions ordered by $\mathcal{E}(q_2(x,z))$ and the set ordered by $\mathcal{E}(q_1(x,z))$ is not nested; neither set is included in the other set. In this case, no unambiguous conclusion can be drawn on the comparison of the ordering power. The following corollary, which summarizes the comparisons among the sets of poverty indices with several commonly used poverty indices as the lower bound, can be readily verified.

**Corollary 4.2.** Consider the following minimum distribution-sensitive poverty indices: the Clark et al. index $q(x, z) = \left(1 - \frac{1}{z} \sum x_i \right)^{\beta}$ with $\beta < 1$, the Watts index $q(x, z) = \ln z - \ln x$, the Chakravarty index $q(x, z) = \left(1 - \frac{1}{z} \sum x_i \right)^{\alpha}$ with $0 < \alpha < 1$, the Foster et al. index $q(x, z) = \left(1 - \frac{1}{z} \sum x_i \right)^{\gamma}$ with $\gamma > 0$, and the CDS index $q(x, z) = e^{\gamma (z - x)} - 1$ with $\gamma > 0$.

(i) The poverty ordering by $\mathcal{E}(q(x,z))$ with $q(x,z)$ being any one of the above indices has strictly more power than the Atkinson second degree stochastic dominance criterion.

(ii) The power of poverty orderings with the Clark et al. index increases as $\beta$ decreases. Specifically, the poverty ordering for $\beta < 0$ has more power than that with the Watts index, which in turns has more power than that with the Chakravarty index.

(iii) The power of poverty orderings with the Foster et al. index increases as $\alpha$ increases. For given values of $\alpha$, $\beta$ and $z$, there is no clear power comparison between the Clark et al. index and the Foster et al. index. However, if $\beta \to -\infty$, then the poverty ordering with the Clark et al. index has more power; if $\alpha \to \infty$, then the poverty ordering with the Foster et al. index has more power.

(iv) For given values of $\alpha$, $\beta$, $\gamma$ and $z$, the poverty ordering with the CDS poverty index does not have a clear power comparison with either the Clark et al. index or the Foster et al. index.
The results of this corollary can also be illustrated graphically via the well-known Kolm's three-person simplex. In Figs. 1a and 1b, we consider a simplex among three persons with a total income of 12 units and the reference distribution is \( h = (2, 3, 7) \) as in Davies and Hoy [8]. For simplicity we consider the poverty line \( z = 12 \) so that everyone lives in poverty. Figure 1a shows the ordering power of all distribution-sensitive poverty indices, i.e., those with \( q(x, z) = z - x \) as the lower bound, and the power of those indices with the Watts index as the lower bound. In the figure, the power of poverty orderings is represented by the shaded area. The lightly shaded area represents the set of distributions rank ordered by all distribution-sensitive poverty indices, while the darkly shaded area indicates the distributions that all poverty indices which are more distribution-sensitive than the Watts index can additionally order. In other words, the darkly shaded area represents the additional power gained by the more restrictive class of poverty indices. Thus the ordering power of poverty can be significantly increased beyond the Atkinson second degree condition. Also, thanks to Proposition 5 below, this increase is unaffected by the setting of the lower bound poverty line if a range of poverty lines is used.

Figure 1b depicts the power of poverty orderings with the Clark et al. index for \( \beta = 1, 0 \) and \(-1\) (note that \( q(x, z) = 1 - \frac{z}{x} \) for \( \beta = 1 \)). Clearly the power of orderings increases as \( \beta \) decreases; when \( \beta \to -\infty \) the whole simplex can be completely rank ordered. For example, the distribution \( A \) in Fig. 1b cannot be ranked with the reference distribution \( h \) by all poverty indices in \( \mathcal{E}(q(x, z)) \) with either \( q(x, z) = \ln z - \ln x \) or \( q(x, z) = 1 - \frac{z}{x} \). But when the class \( \mathcal{E}(q(x, z)) \) with \( q(x, z) = -\left[1 - \left(\frac{z}{x}\right)^{-1}\right] \) is considered, \( A \) and \( h \) can be ranked: \( A \) has more poverty than \( h \) for all poverty indices in \( \mathcal{E}(q(x, z)) \). Thus, for any two distributions within this simplex, one can always find a class of minimum distribution-sensitive poverty indices to unanimously rank these distributions. The ordering power of \( \mathcal{E}(q(x, z)) \) with \( q(x, z) \) being the Foster et al. index and the CDS index can also be illustrated in an analogous manner.

3.3. Poverty Orderings at a Lower Poverty Line

An interesting and important property of the Atkinson condition, as stated in Corollary 2.1, is that the dominance relation at a given poverty line will also prevail at any lower poverty line. That is, if \( P(X; z_2) \succeq P(Y; z_1) \) for all poverty indices \( P \) in \( \mathcal{E}(q) \) with \( q(x, z) = z - x \), then \( P(X; z_2) \succeq P(Y; z_2) \) holds for all \( z_2 \in (0, z_1) \). This property is made more appealing by the fact that second degree stochastic dominance relation over \((0, z_1)\) is also the necessary and sufficient condition for the poverty gap \((q(x, z) = z - x)\) to indicate no less poverty in \( X \) than in \( Y \) for all poverty lines \( z \in (0, z_1) \) (Foster and Shorrocks [13, 14]). Thus, to some extent, the two branches of poverty orderings, those for a class of poverty indices and
FIG. 1a. Poverty orderings for $\Xi(q)$ with $q = z - x$ and $q = \ln z - \ln x$. 

those for all poverty lines, can be unified. A natural question to explore here is whether this property can be extended to poverty orderings by $\Xi(q(x, z))$ with $q(x, z)$ being strictly more distribution-sensitive than $z - x$.

The following proposition describes the condition on $q(x, z)$ under which this property can be carried over.

**Proposition 5.** Consider two poverty lines $z_2 < z_1$ and any distributions $X \in \Psi^n$ and $Y \in \Psi^n$. The necessary and sufficient condition for $P(X; z_1) \geq P(Y; z_1)$ for all $P \in \Xi(q(x, z))$ to imply $P(X; z_2) \geq P(Y; z_2)$ for all $P \in \Xi(q(x, z))$ is that $q(x, z_2)$ is a nondecreasing and convex transformation of $q(x, z_1)$.

**Proof.** Use (3.2), the above condition can be restated as: if and only if $q(x, z_2)$ is a nondecreasing and convex transformation of $q(x, z_1)$ does \( \sum_{i=1}^{k} q(x_i, z_2) \geq \sum_{i=1}^{k} q(y_i, z_1) \) for $k = 1, 2, \ldots, n$ necessarily imply \( \sum_{i=1}^{k} q(x_i, z_2) \geq \sum_{i=1}^{k} q(y_i, z_2) \), $k = 1, 2, \ldots, n$, for any distributions $X$ and $Y$ in $\Psi^n$.

Clearly, $q(x, z_2)$ is a continuous and nondecreasing function of $q(x, z_1)$. This is because if $q(x, z_1)$ increases for a fixed $z_1$, then $x$ must decrease and $q(x, z_2)$ will increase if $x < z_2$ and remain unchanged if $x > z_2$. Next,
q(x, z₂) can be regarded as a transformation of q(x, z₁). Specifically, q(x, z₁) is transformed into q(x, z₂) for x ≤ z₂ and into zero if x > z₂. Thus the same argument used in proving Proposition 4 can be used to complete the proof of this proposition.

The condition imposed on q(x, z) that q(x, z₂) is a convex transformation of q(x, z₁) is satisfied by all poverty indices discussed in this paper. In particular, if q(x, z) is a Dalton-type, say, q(x, z) = q(z) − q(x), then q(x, z₂) = q(x, z₁) + [q(z₂) − q(z₁)] for x ≤ z₂ and zero otherwise. Hence, the Clark et al. index, the Watts index, the Chakravarty index and the CDS index all possess the property described in Proposition 5. For the Foster et al. index, q(x, z₂) = (z₁/z₂) * [z₂/z₁ − 1 + [q(x, z₁)z²]²] for x ≤ z₂, hence q(x, z₂) can be verified to be a strictly convex transformation of q(x, z₁) for x ≤ z₂.

But can the condition that q(x, z₂) is a convex transformation of q(x, z₁) be justified for a general poverty index? Consider consecutive decreases in x from x to x − σ and from x − σ to x − 2σ. The condition says that both q(x, z₂) and q(x, z₁) will increase but q(x, z₂) may do so at a faster (at least not slower) rate than q(x, z₁), i.e.,

\[ \frac{q(x - 2\sigma, z_2) - q(x - \sigma, z_2)}{q(x - \sigma, z_2) - q(x, z_2)} \geq \frac{q(x - 2\sigma, z_1) - q(x - \sigma, z_1)}{q(x - \sigma, z_1) - q(x, z_1)} \]
though the absolute increase in \( q(x, z) \) may not exceed that in \( q(x, z_2) \). Of all existent poverty axioms, no single axiom or any combination of them may imply the above condition. Mathematically, the condition amounts to requiring that the measure of distribution-sensitivity, \( s_q(x, z) \), be a decreasing function of \( z \), that is, \( \partial s_q(x, z)/\partial z \leq 0 \). Expanding this requirement, we obtain \( \partial q(1)(x, z)/\partial z \leq \partial q_1(x, z)/\partial z \). Clearly, the assumptions we have made on \( q(x, z) \) do not yield this condition; the only axiom that has anything to do with the behavior of the poverty line is the increasing poverty line axiom which only entails \( q_1(x, z) > 0 \) for \( x < z \). Although there is no compelling reason to justify this requirement as an axiom, there seems no strong ground to dismiss it either. In addition, as we have seen, all existing poverty indices possess this property. Thus we may regard the result presented in Proposition 5 as holding in general, though a pathological example may be constructed where \( q(x, z_2) \) is not a convex transformation of \( q(x, z_1) \).

Finally, it is interesting to examine the relation between the ordering condition for all indices in the class \( \mathbb{E}(q(x, z)) \) and the ordering condition of the poverty index \( Q(X; z) = \frac{1}{n} \sum_{i=1}^{n} q(x_i, z) \) for all poverty lines. The following proposition indicates that in general the ordering condition for all poverty indices in \( \mathbb{E}(q(x, z)) \) is stronger than the ordering condition of \( Q(X; z) \) for all poverty lines.

**Proposition 6.** If \( P(X; z_1) \geq P(Y; z_1) \) for all poverty indices \( P \) in \( \mathbb{E}(q) \), and \( q(x, z) \) satisfies the condition specified in Proposition 5, then \( \frac{1}{n} \sum_{i=1}^{n} q(x_i, z_2) \geq \frac{1}{n} \sum_{i=1}^{n} q(y_i, z_2) \) for all \( z_2 \in (0, z_1] \).

**Proof.** Recall that the poverty ordering condition (3.2) at \( z_1 \) is \( \sum_{i=1}^{n} q(x_i, z_1) \geq \sum_{i=1}^{n} q(y_i, z_1) \) for \( k = 1, 2, \ldots, n \). Thus if \( q(x, z_2) \) is a convex transformation of \( q(x, z_1) \) for \( z_2 \leq z_1 \), Proposition 5 entails that \( \frac{1}{n} \sum_{i=1}^{n} q(x_i, z_2) \geq \frac{1}{n} \sum_{i=1}^{n} q(y_i, z_2) \) for all \( z_2 \in (0, z_1] \).

The converse is generally not true. For example, for the Foster et al. index \( q(x, z) = (1 - x/z)^{a} \), it is easy to show that the poverty-line condition does not imply the poverty-index condition.\(^{20}\) For a Dalton-type poverty

\(^{20}\) Consider, for example, the following two distributions \( X = \{1, 3, 5, 7, 9\} \) and \( Y = \{1.5, 2.5, 5, 6.5, 9.5\} \). Since \( Y \) is obtained from \( X \) by a “favorable composite transfer” (Shorrocks and Foster [32]): transferring 0.5 from person 2 to person 1 and from person 4 to person 5, by Lemma 4 and Theorem 3 of Foster and Shorrocks [14], \( \frac{1}{n} \sum_{i=1}^{n} (1 - y_i/z)^{a} \geq \frac{1}{n} \sum_{i=1}^{n} (1 - x_i/z)^{a} \) for all \( z \in \mathbb{R} \). However, for \( z = 10 \), it is easy to verify that condition (3.2) does not hold.
index \( q(x, z) \), however, these two conditions are the same as shown by Foster and Jin [11]. It would be interesting to know whether the Dalton-type indices are the only class that possesses this property.

4. CONCLUDING REMARKS

We have introduced in this paper a measure of distribution-sensitivity for a poverty index. The measure resembles the Arrow-Pratt measure of risk aversion. Unlike the measure of risk aversion, the measure of poverty distribution-sensitivity itself has a straightforward interpretation: it quantifies the relative change in poverty as a result of income transfers. Also a poverty index is more distribution-sensitive than another index if and only if the former is an increasing and convex transformation of the latter. We have also examined the notion of poverty aversion and pointed out that poverty aversion and distribution-sensitivity may mean the same thing. Thus the measure of distribution-sensitivity can also serve as a measure for poverty aversion.

We then examined the implication of the measure of distribution-sensitivity for poverty orderings. When poverty indices are restricted to those possessing a minimum nonzero distribution-sensitivity, our ordering condition is a generalization of (poverty) deprivation profile dominance that has been characterized by Spencer and Fisher, Jenkins and Lambert, and Shorrocks. Compared with the Atkinson second degree ordering condition, our condition is more complete in ranking distributions as shown in Kolm’s three-person simplex. Various useful properties of the new ordering conditions have also been examined and discussed.

While we imposed a nonzero lower bound on the distribution-sensitivity of a poverty index to avoid poverty indices which are “too close” to the poverty gap ratio, we may also want to avoid poverty indices that have “too much” distribution-sensitivity. For example, the class of poverty indices that we considered in Section 3 clearly includes the Rawlsian maximin poverty indices. Again the transfer axiom does not specify a meaningful upper boundary of sensitivity for a poverty index. Following Meyer [22], we can also explore poverty orderings for poverty indices with maximum distribution-sensitivity, that is, the class

\[
\Pi(q(x, z)) := \left\{ P(X; z) = \frac{1}{n} \sum_{i=1}^{n} p(x_i, z) \left| s_d(x, z) \leq s_d(x, z) \text{ for all } x \in [0, z] \right. \right\}
\]

(4.1)
for a pre-specified poverty deprivation function \( q(x, z) \). It is easy to see that each \( p(x, z) \) is a continuous, increasing and concave function of \( q(x, z) \). Using Proposition B.2 of Marshall and Olkin [20, p. 109],\(^{21}\) we know that the necessary and sufficient condition for \( P(X; z) \geq P(Y; z) \) for all \( P \) in \( \Pi(q(x, z)) \) is that \( \{q(x_i, z)\} \) is weakly supermajorized by \( \{q(y_i, z)\} \). That is, the ordering condition is

\[
\sum_{i=k}^{n} q(x_i, z) \geq \sum_{i=k}^{n} q(y_i, z), \quad k = n, n-1, ..., 1. \tag{4.2}
\]

Results similar to Propositions 4 through 6 can also be similarly derived.

Compared with \( \mathcal{E}(q) \) which contains some extremely sensitive poverty indices, the class \( \Pi(q) \) goes to another extreme: it contains indices that violate the transfer axiom. In this regard, the above extension may not be very interesting. Ideally, one would like to bound poverty indices from both ends, that is, to impose both upper and lower limits on distribution-sensitivity. Unfortunately, as shown by Meyer [21], there is no closed form solution to this ordering problem. Meyer solved this type of problem in the setting of an optimal control framework and provided the calculating rule. It is useful and possible to extend Meyer’s method to poverty orderings.

Having characterized \( s(x, z) \) as the measure of distribution-sensitivity for additively separable poverty indices, we may also want to apply the same idea to other poverty indices such as the Sen [28] and Thon [34] indices. Consider, for example, the Sen index and two poor persons \( i \) and \( j \) (\( i < j \)). The (relative) marginal reduction in poverty for a small rank-preserving increment of income to \( i \) rather than to \( j \) is \( \frac{j-i}{q} \), where \( q \) is the number of the poor. For the Thon index, the (relative) marginal reduction in poverty is \( \frac{j-i}{q+1} \). Although \( \frac{j-i}{q+1} \geq \frac{j-i}{q} \), it is questionable to say that the Sen index is more distribution-sensitive than the Thon index, which is not a continuous function of the Sen index. It would thus be useful to develop a measure of distribution-sensitivity (poverty aversion) for these additively non-separable poverty indices.

Finally, one may also be interested in deriving ordering conditions for the class of poverty indices with diminishing poverty aversion (DPA). The Clark et al. indices belong to this class. This practice mirrors Vickson’s [36, 37] investigation of stochastic dominance for the DARA (decreasing absolute risk aversion) utility functions. In general, one may expect that the ordering power of the DPA class of poverty indices would be greater than the Atkinson second degree condition.

\(^{21}\) Note that if \( g(\cdot) \) is increasing and convex then \( -g(\cdot) \) is decreasing and concave. Thus the proposition is applicable to our case.
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