ON THE MEASUREMENT OF THE DEGREE OF PROGRESSION

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It is shown that residual progression (elasticity of income after tax with respect to income before tax) is the measure of tax progression most closely connected with the redistributive effect of the tax system, judged by the criterion of Lorenz-domination. It is also shown that if and only if the tax schedule is one of constant residual progression, will the redistributive effect of the tax system be unaffected by a proportional increase of all incomes before tax.

1. Introduction

It is generally agreed that a progressive tax system should be defined as one where the average rate of taxation increases with income before tax. The degree of progression, however, is often referred to by politicians and economists with no precise meaning attached to it.

The ambiguity of the latter concept was discussed by Musgrave and Tun Thin (M–T) (1948) in their well-known article ‘Income tax progression 1929–1948’. They suggested the following four local measures of progression:

(1) Average rate progression (the derivative of the tax rate with respect to income before tax);
(2) Marginal rate progression (the derivative of the marginal tax rate with respect to income before tax);
(3) Liability progression (elasticity of tax liability with respect to income before tax);
(4) Residual income progression (elasticity of income after tax with respect to income before tax).

These measures are all compatible with the basic definition of a progressive tax system. Any progressive tax is by each measure considered ‘more progressive’ than a proportional tax.

As could be expected, the different measures all had different stories to tell.

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about the development of tax progression in the U.S. M–T could also contend that it was not possible, on the grounds of any of the sacrifice formulae, to single out one measure of progression as the 'correct' one.

Today it seems natural to choose income redistribution instead of traditional equity theory as a framework for a discussion of the degree of progression. Recent work on the measurement of income inequality has provided strong justification for the use of the Lorenz criterion when ranking income distributions with respect to income inequality.\(^1\) When the income distribution before tax is given, this criterion could also be used to decide whether one tax system is more redistributive than another. Suppose that two tax schedules give rise to income distributions after tax with nonintersecting Lorenz-curves, then the tax schedule related to the dominated Lorenz-curve can be considered unambiguously more redistributive than the other.

The purpose of this note is to show that as soon as the context chosen is income redistribution judged by the Lorenz criterion, there is just one logical measure of progression. The argument rests on the following reasonable requirement for a local measure of progression: If one tax system is everywhere, according to the measure, more progressive than the other, then it should also be unambiguously more redistributive than the other.

It will be shown that the only measure to meet this requirement is the *elasticity of income after tax with respect to income before tax* or, in the M–T terminology, the *residual progression*. This relation between the global concept of redistribution and the proposed local measure of progression is of analytical value, and is also very useful in the practical work of assessing alternative tax systems.

Generally, the redistributive effect of a tax system is affected by a change in the income distribution before tax. Empirical work indicates that while the general level of nominal income increases over time, there is a considerable stability in the relative distribution of personal income. Therefore it is of interest to study the redistributive effects from a mere change of the scale of the income distribution before tax. For a tax schedule with constant residual progression, tax redistribution is unaffected by any proportional change in income distribution before tax. It will also be shown that it is the only tax schedule with this property. The paper ends with a short discussion of the relation between residual progression and other characteristics of the tax function.

### 2. Notation and definitions

Frequent use will be made of the following set of definitions, all referring to the tax function on the individual level:

\[
y = \text{income before tax},
\]

s(y) = amount of tax paid (s is a function of y, where the form of the function is specified in the tax laws),

x(y) = y - s(y) = income after tax,

t(y) = s(y)/y = average tax rate,

M(y) = ds/dy = marginal tax rate,

a(y) = (dx/dy)(y/x) = residual progression (elasticity of income after tax),

e(y) = (ds/dy)(y/s) = liability progression (tax elasticity).

The two other M-T measures of progression, that will not be discussed further, are dt/dy (= average rate progression) and dM/dy (= marginal rate progression).

3. Progression and Lorenz-domination

In order to decide whether a particular income distribution is more equal than another, the concept of Lorenz-domination (LD) will be used. According to this criterion, an income distribution is more equal than another if and only if its Lorenz-curve lies completely inside the Lorenz-curve of the other. The criterion has a long tradition in the study of income inequality. It can moreover be shown that it is equivalent to the intuitively appealing 'principle of transfers'. LD provides a partial ordering on the set of income distributions. Each one of the conventional measures of inequality gives a particular total ordering on the same set. It can be shown that most of these orderings are in accordance with LD. A further justification for the use of LD in this context is provided by the corollary of this section.

In this section and the following one, we will work with discrete representations of income distributions. An income distribution is thus given by a vector \( p = (p_1, \ldots, p_n) \), \( p_1 \leq p_2, \ldots, \leq p_i, \ldots, \leq p_n \), where \( p_i \) is the income of the \( i \)th person. The Lorenz-curve of a particular distribution is given by the curve defined by the following set of ordered pairs,

\[
\left( \frac{v}{n}; \frac{1}{n} \sum_{i=1}^{v} p_i \bigg/ \frac{1}{n} \sum_{i=1}^{n} p_i \right), \quad v = 1, \ldots, n.
\]

A formal definition of LD is now: The vector \( p \) is Lorenz-dominated by the vector \( q \) if and only if


\(^3\)See Dalton (1920).


\(^5\)See Kolm (1969).
\[
\sum_{i=1}^{\nu} p_i \leq \sum_{i=1}^{\nu} q_i, \quad \nu = 1, \ldots, n, \tag{1}
\]

with equality for \( \nu = n \).

The following lemma is essential for the proposition in this section.

**Lemma.** Consider two vectors \( p = (p_1, \ldots, p_n) \) and \( q = (q_1, \ldots, q_n) \), where \( 0 \leq p_1 \leq p_2 \leq \ldots \leq p_n \), and \( 0 \leq q_1 \leq q_2 \leq \ldots \leq q_n \). If for \( 1 < i \leq n \),

\[
q_i/q_{i-1} < p_i/p_{i-1},
\]

then the vector \( p \) is Lorenz-dominated by the vector \( q \).

**Proof.** The inequality (2) implies that

\[
\sum_{i=1}^{n} (q_i/q_1) < \sum_{i=1}^{n} (p_i/p_1)
\]

and

\[
\sum_{i=1}^{n} (q_i/q_n) > \sum_{i=1}^{n} (p_i/p_n).
\]

Since both vectors are positive, we then have

\[
q_i \left/ \sum_{i=1}^{n} q_i \right. > p_i \left/ \sum_{i=1}^{n} p_i \right.
\]

and

\[
q_n \left/ \sum_{i=1}^{n} q_i \right. < p_n \left/ \sum_{i=1}^{n} p_i \right.
\]

Both \( p \) and \( q \) are arranged in increasing order so there exists an integer \( \nu' \) such that

\[
q_i \left/ \sum_{i=1}^{n} q_i \right. \geq p_i \left/ \sum_{i=1}^{n} p_i \right., \quad \text{where } i \leq \nu',
\]

and

\( ^6 \text{Obviously the reverse is not true. Let } p = (1; 2; 3; 16) \text{ and } q = (3; 4; 7; 8); p \text{ is LD by } q, \text{ but (2) is not always satisfied.} \)
\[ q_i \sum_{i=1}^{n} q_i < p_i \sum_{i=1}^{n} p_i, \quad \text{where } i > v'. \] (4)

From the identity

\[ \sum_{i=1}^{n} q_i \sum_{i=1}^{n} q_i = \sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i \]

and (4), we have

\[ \sum_{i=1}^{n} q_i \sum_{i=1}^{n} q_i \geq \sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i, \quad v > v'. \]

The rest of (1) follows directly from (3).

**Proposition 1.** Consider two tax schedules with residual progressions given by \( a_1(y) \) and \( a_2(y) \). The schedules are working on the income distributions \( x_1(y_1, \ldots, y_n) \) (\( v = 1, 2 \), respectively), where \( \alpha_v \) is a scalar factor that may or may not change its value with \( v \). The resulting distributions of income after tax are given by \( x_1 \) and \( x_2 \), respectively. (i) If \( a_1(y) < a_2(y) \) everywhere, then \( x_2 \) is LD by \( x_1 \). (ii) A necessary condition for \( x_2 \) to be LD by \( x_1 \) for any \( y \) is that \( a_1(y) < a_2(y) \) everywhere.

**Proof.** (i) It follows by the definition of elasticity of income after tax that when \( a_1(y) < a_2(y) \) everywhere, then \( x_1/x_1 < x_2/x_2 \). The first part of the proposition now follows from the Lemma.

(ii) If \( a_1(y) < a_2(y) \), except for one interval where \( a_1(y) > a_2(y) \), we could always choose an income distribution before tax that lies completely within the latter interval.

As a corollary to this proposition we get an interesting analogy between LD and the basic definition of progression.

**Corollary.** If and only if a tax system is progressive everywhere, will any income distribution before tax be Lorenz-dominated by the resulting income distribution after tax.

**4. Constant residual progression**

The Lorenz-curve of an income distribution is not affected by the scale of incomes; in other words, a proportionate change of all incomes leaves the Lorenz-curve of the distribution intact.

It is easily verified that a tax schedule of constant residual progression has the property to preserve the Lorenz-curve for the distribution of incomes after tax,
when the scale of all incomes is changed. In the following proposition it is also shown that it is the only tax schedule with this property. That is, in any other tax system there is a change in the redistributive effect resulting from, e.g., an inflationary proportional increase of all incomes before tax.

**Proposition 2.** The redistributive effect of a particular tax schedule is unaffected by a proportionate change in all incomes before tax if and only if \( a(y) \) is constant (i.e., \( x = by^a \), where \( y \) is income before tax and \( x \) is income after tax).

**Proof.** (i) The sufficiency part is trivial.

(ii) To prove the necessity, let the relation between income before tax \( y \) and income after tax \( x \) be given by \( x = f(y) \). If the before tax distribution with a scale factor is given by \( ay = (ay_1, \ldots, ay_n) \), we then get the after tax distribution \( x = (f(ay_1), f(ay_2), \ldots, f(ay_n)) \). It is easily checked that the requirement that the Lorenz-curve of \( a \) is invariant for changes in \( x \) implies that for any pair of elements \( y_i, y_j \), the ratio \( f(ay_i)/f(ay_j) \) is a function of \( y_i \) and \( y_j \) only, i.e.,

\[
\frac{f(ay_i)}{f(ay_j)} = G(y_i; y_j).
\]

For \( y_j = 1 \) we get

\[
f(ay_i) = f(a)y_i.
\]

There is a well-known theorem first proved by Cauchy (1821) that the only function to satisfy this equation is \( x = by^a \).

5. The relation to other characteristics of the tax function

The measure of progression \( a \) advocated here can be written as a function of the individual's tax rate \( t \) and marginal tax rate \( M \) in the following way,

\[
a(y) = (1 - M(y))(1 - t(y)). \tag{5}
\]

So \( a \) could be lowered either by an increase in the marginal tax rate or a decrease in the average tax rate. If, for a given revenue constraint on the macro level, we consider a uniform increase in the rate of progression, we would expect people with low incomes to get their tax rates reduced while rates would be raised for high-income people. The latter category must consequently also get their marginal tax rates increased, while there would be a possibility for low income people to get marginal tax rates reduced.

An overall increase in residual progression will therefore generally result in one group (low income earners) getting their marginal tax rates as well as their tax rates decreased, one group (middle income earners) having their marginal
tax rates increased and their tax rates decreased, and one group (high income earners) that gets their tax rates as well as their marginal tax rates increased. The effect on work effort of an increased residual progression will therefore, according to conventional analysis, be undetermined in the low income group and the high income group. In the middle, however, the income effect and the substitution effect will both work for a diminished labour supply.\(^7\) [See Musgrave (1959, pp. 243–245).]

Finally, something should be said about the concept of liability progression, which seems to be the measure of progression most frequently used. There is the same relation between this measure and the distribution of the tax burden as there is between residual progression and the distribution of income after tax. If we start from a vector of income before tax \(y\), two particular tax systems give the vectors \(S^1\) and \(S^2\) of the tax liabilities. If the Lorenz-curve of \(S^1\) lies completely inside that of \(S^2\), we may say that the tax burden of system 1 is unambiguously more evenly distributed than that of system 2. A more even distribution in this case means of course that a larger share of the tax burden falls on the lower income groups. The following proposition, that is analogous to proposition 1, gives the relation between liability progression and the distribution of the tax burden.

**Proposition 3.** Consider two tax schedules with liability progressions given by \(e_1(y)\) and \(e_2(y)\). The schedules are working on the income distributions \(x_v(y_1, \ldots, y_n)\) (\(v = 1, 2\), respectively), where \(x_v\) is a scalar factor that may or may not change its value with \(v\). The resulting distributions of the tax burdens are given by \(S^1\) and \(S^2\), respectively. (i) If \(e_1(y) < e_2(y)\) everywhere, then \(S^2\) is LD by \(S^1\). (ii) A necessary condition for \(S^2\) to be LD by \(S^1\) for any \(y\), is that \(e_1(y) < e_2(y)\) everywhere.

**Proof.** The proposition is proved in exactly the same way as proposition 1.

For a given average tax \(t\), the two measures vary inversely.\(^8\) More interesting is that when the average tax for a given income is increased, residual progression kept constant, liability progression will be diminished. To see this we rewrite the definition of the two measures,

\[
a(p, y) = (1 - M(p, y))(1 - t(p, y)),
\]

\[
e(p, y) = M(p, y)/t(p, y),
\]

\(^7\) A utility function of the Cobb-Douglas type provides an interesting special case. Each individual then maximizes his utility \(u = C((24 - H)^{1-a})\), subject to the constraint \(C = f(wH)\), where \(C = \text{consumption of goods}, H = \text{hours worked}, w = \text{wage rate}, \) and \(f\) represents the function from income before tax to income after tax. The maximizing procedure gives a labour supply \(H = 24 ak/(1 + ak)\), where \(k = a/(1 - a)\), and \(a = f'((wH)wH)/f(wH) = \text{residual progression}\). So when the parameters of the utility function are given, \(H\) depends only on residual progression. An increased residual progression \((da > 0)\) will here always diminish labour supply \((dH/da > 0)\).

\(^8\) From the definitions, we have \(a = (1 - et)/(1 - t)\).
where the tax characteristics depend on the public parameter \( p \) and income before tax \( y \). Now consider a change in the tax laws such that \( \partial M/\partial p > 0 \), \( \partial t/\partial p > 0 \) and \( \partial a/\partial p = 0 \).

From the first expression in (6) we find that

\[
\frac{\partial a}{\partial p} = 0 \Rightarrow \frac{\partial t}{\partial p}/(\partial M/\partial p) = (1 - t)/(1 - M).
\]

We also have

\[
\frac{\partial e}{\partial p} = \frac{\partial t}{\partial p} - M \cdot \frac{\partial t}{\partial p} |t^2. \]

When the tax system is progressive, \( M > t \). Therefore, by (7) and (8), \( \partial e/\partial p < 0 \).

So a general increase in the tax burden, with tax redistribution kept constant, will increase the share of the tax burden borne by the low income groups.

References


