SINGLE-CROSSING CONDITIONS IN COMPARISONS OF TAX PROGRESSIVITY

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Statements to the effect that one tax is more progressive than another are frequently made, but seldom explained. Starting from a criterion of Lorenz domination, this paper shows how an intuitively appealing single-crossing condition on appropriately normalized tax schedules may be used in progressivity comparisons. This condition is closely related to, and illuminates, the necessary and sufficient condition derived by Jakobsson (1976) for the distributional implications of alternative taxes to be inferred from their schedules alone.

1. Introduction

In discussions of tax policy one frequently finds one tax described as 'more progressive' than another. Sometimes the basis of the progressivity comparison is clearly specified, but more usually it is not. Since the assessment of relative progressivity typically revolves around the distributional implications of taxation, it is the basis of the underlying distributional judgements which remains vague. The concept of Lorenz domination underlies the approach to progressivity comparisons adopted in this paper. [See Atkinson (1970) and Sen (1973) for discussion of the welfare implications and other aspects of the Lorenz criterion for inequality comparisons.] One tax will be described as 'more progressive' than another for a given distribution of pre-tax income if and only if the Lorenz curve of the distribution of post-tax income to which the more progressive tax gives rise lies everywhere on or inside that associated with the less progressive tax. If and only if one tax is more progressive than another for all distributions of pre-tax income, it will be described as 'uniformly more progressive'.

The purpose of this paper is to describe the use of a single-crossing condition on suitably normalized tax schedules in progressivity comparisons, and to suggest a general framework for statements on relative progressivity. The single-crossing condition is discussed in section 2, where it is shown that

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the condition may be used to infer a partial ordering of distributions of post-tax income by the Lorenz criterion. To this partial ordering will correspond a ranking of taxes by progressivity.¹ The relationship between the single-crossing condition and the ranking of taxes by uniform progressivity is explained in section 3. Examples are given to suggest the usefulness of the concepts considered in the paper.

2. Single-crossing

We consider the comparison between two income taxes, \( T_M \) and \( T_L \), the convention being that \( T_M \) is more progressive than \( T_L \). Each tax is completely characterized by a strictly positive, continuous and non-decreasing function \( n_i, i = M, L \), giving post-tax income as a function of pre-tax income. We assume that pre-tax income is distributed over some interval \( Y = [y, \bar{y}] \), and denote by \( \mathcal{A} \) the set of distribution functions on \( Y \). The distribution function for post-tax income induced by \( T_i, i = M, L \), conditional on pre-tax distribution \( G \in \mathcal{A} \) is denoted \( F_i | G \). The distribution of pre-tax income is assumed to be independent of the tax system in operation.

Formalizing the definition given in the introduction, we shall say that \( T_M \) is 'more progressive' than \( T_L \) for \( G \) if and only if \( F_M | G \) weakly Lorenz dominates \( F_L | G \).² Our assumptions on \( n_i \) then imply that \( T_M \) is more progressive than \( T_L \) for \( G \) if and only if

\[
\int_y^\bar{y} n_M(y) dG(y) \leq \int_y^\bar{y} n_L(y) dG(y), \quad \forall G \in \mathcal{A}.
\]

We shall also say that a function \( h_1 \) 'single-crosses' \( h_2 \) on \( Y \) if and only if there exists some \( y^* \) such that

\[
h_1(y) \geq h_2(y), \quad \forall y \in [y, y^*],
\]

\[
h_1(y) \leq h_2(y), \quad \forall y \in [y^*, \bar{y}].
\]

In discussing the single-crossing condition it is helpful to distinguish between equal-yield and non-equal-yield comparisons.

¹Although progressivity statements will be related only to the distribution of post-tax income, the arguments presented below are readily adapted for the purposes of an alternative approach, advocated by Kakwani (1977), relating progressivity to the distribution of the tax burden.

²We work in terms of weak Lorenz domination throughout the paper, and all subsequent references to Lorenz dominance are to be interpreted in that sense. It is straightforward, but cumbersome, to reformulate the discussion in terms of strict Lorenz domination.
2.1. Equal-yield comparisons

If $G$ is such that $T_M$ and $T_L$ raise the same revenue, then

$$\int \{ n_M(y) - n_L(y) \} \, dG(y) = 0$$

so that (1) reduces to the condition:

$$\int_Y \{ n_M(y) - n_L(y) \} \, dG(y) \geq 0, \quad \forall v \in Y.$$  (4)

Denote by $A(M, L)$ that subset of $A$ for which (3) holds, i.e. the set of distributions of pre-tax income for which $T_M$ and $T_L$ raise the same revenue. Then:

**Proposition 1.** $T_M$ is more progressive than $T_L$ if and only if $n_M$ single-crosses $n_L$ on $Y$.

**Proof.**

(i) Sufficiency. For $v \in [y, y^*]$ (4) follows directly from the definition of single-crossing. For $v \in [y^*, y]$:

$$\int_Y \{ n_M(y) - n_L(y) \} \, dG(y)$$

$$= \int_Y \{ n_M(y) - n_L(y) \} \, dG(y) + \int_v^y \{ n_L(y) - n_M(y) \} \, dG(y) \geq 0$$

from (3) and the definition of single-crossing.

(ii) Necessity. Suppose, for a contradiction, that $n_M$ and $n_L$ cross more than once on $Y$. Then there exists at least one subinterval on which $n_M$ single-crosses $n_L$ and at least one on which the reverse holds. By (i), it therefore suffices to show that if $n_M$ single-crosses $n_L$ on any subinterval $\tilde{Y}$ of $Y$, then there exists $\tilde{G} \in A$ such that $T_M$ and $T_L$ raise the same revenue on $\tilde{G}$. This, however, follows readily from the assumed continuity of $n_i, i = L, M$.

The implication of proposition 1 is that if $T_M$ and $T_L$ are observed to raise the same revenue with $n_M$ single-crossing $n_L$, then $T_M$ might reasonably be described as more progressive than $T_L$ given the underlying distribution of pre-tax income. Such a description has much intuitive appeal: relative to $T_L$, $T_M$ increases the net income of all those with pre-tax income below a certain level whilst correspondingly reducing the net incomes of those with pre-tax income above that level, benefiting an identifiably 'poor' group at the expense of an identifiably 'rich' one. Conversely, if $n_M$ and $n_L$ cross more than
Once, as in fig. 1, then there exist distributions of pre-tax income across which the progressivity ranking is reversed. For any \( \hat{y} \in (y_1, y_2) \) there exist two distributions of pre-tax income, one confined to \([\hat{y}, \hat{y}]\) and the other to \([\hat{y}, \hat{y}]\), such that \( T_M \) and \( T_L \) raise the same revenue, but then \( T_M \) is more progressive than \( T_L \) for the former distribution and less progressive for the latter.

2.2. Non-equal-yield comparisons

In equal-yield comparisons the graphs of \( n_M \) and \( n_L \) must intersect. This need not be the case when the taxes under consideration raise different revenues. However, it is straightforward to extend the above argument to encompass non-equal-yield comparisons.

We may rewrite (1), the condition for Lorenz domination of \( F_L \mid G \) by \( F_M \mid G \), as

\[
\frac{\int \n_M^G(y) \, dG(y)}{\int \n_L^G(y) \, dG(y)} \geq \frac{\int \n_M^G(y) \, dG(y)}{\int \n_L^G(y) \, dG(y)}, \quad \forall v \in Y,
\]

where
denotes the share of total net income associated with pre-tax income $y$ under $T_i$ given that the distribution of pre-tax income is $G$. Thus, the tax schedules have been normalized in such a way that

$$\int n_i^G(y) \, dG(y) = 1, \quad i = L, M,$$

enabling us to transform a non-equal-yield comparison between the income schedules $n_M$ and $n_L$ into an equal-yield comparison between the surrogate schedules $n_M^G$ and $n_L^G$. It then follows from the sufficiency part of proposition 1 that $T_M$ is more progressive than $T_L$ for $G$ if $n_M^G$ single-crosses $n_L^G$ on $Y$. This result also has substantial intuitive appeal: it is sufficient for $T_M$ to be more progressive for $G$ than $T_L$ that there exist some level of pre-tax income such that the share in total net income of all those with incomes below (above) that level is higher (lower) under $T_M$ than under $T_L$.

Now consider the problem of making progressivity comparisons between more than two tax schedules. As is well known, single-crossing is not transitive (though it is acyclic). In the present context, however, the implications of this observation are limited, since Lorenz domination is transitive. Suppose, for instance, that $n_M^G$ single-crosses $n_L^G$ and that $n_L^G$ single-crosses $n_K^G$. Then, although we cannot be sure that $n_M^G$ single-crosses $n_K^G$, we may infer that $F_M \mid G$ Lorenz dominates $F_K \mid G$ and hence that $T_M$ is more progressive than $T_K$ for $G$. Thus, for a given distribution of pre-tax income, the single-crossing condition may be used to construct a partial ordering of tax systems by progressivity.

Single-crossing has received little attention, if any, in the progressivity literature. We find this neglect surprising. The single-crossing condition described above is exceedingly straightforward. It enables quite powerful statements concerning the distributional implications of different tax systems to be made on the basis of information on tax schedule specifications and total net income alone. It would appear to be particularly valuable when income distribution data are incomplete or of low quality. Not least, single-crossing seems to correspond closely to popular views on the nature and purpose of progressivity comparisons.

3. Single-crossing and uniform progressivity

We shall say that $T_M$ is ‘uniformly more progressive’ than $T_L$ if and only if $F_M \mid G$ Lorenz dominates $F_L \mid G \ \forall G \in A$. Thus $T_M$ is uniformly more progressive than $T_L$ if and only if it is more progressive than $T_L$ whatever
the distribution of pre-tax income. Denote by $\phi_i$ the elasticity of post-tax income with respect to pre-tax income under $T_i$; i.e. $\phi_i(y) = n'_i(y)/n_i(y)$, a prime indicating differentiation. (We assume throughout this section that the tax schedules under consideration are continuously differentiable.) Then we have:

**Proposition 2 [Jakobsson (1976)].** $T_M$ is uniformly more progressive than $T_L$ if and only if $\phi_M(y) \leq \phi_L(y) \forall y \in Y$.

Jakobsson's proposition is an extremely valuable one, providing a complete characterization of the circumstances in which the distributional implications of alternative taxes (in terms of Lorenz domination) may be inferred from the specification of tax schedules alone. However, the restriction on net income schedules implied by the condition is a stringent one, so that it is important to establish the relationship between the circumstances in which relative progressivity statements may be made on the basis of Jakobsson's condition and those in which they may be made on the basis of the single-crossing condition. This we do in:

**Proposition 3.** $T_M$ is uniformly more progressive than $T_L$ if and only if $n_M^G$ single-crosses $n_L^G$ on $Y \forall G \in \Lambda$.

**Proof.**

(i) Sufficiency. If $n_M^G$ single-crosses $n_L^G \forall G \in \Lambda$ then by (i) of proposition 1, $F_M|G$ Lorenz dominates $F_L|G \forall G \in \Lambda$. Thus, from the definition, $T_M$ is uniformly more progressive than $T_L$.

(ii) Necessity. It suffices to show that if $\phi_M(y) \leq \phi_L(y) \forall y \in Y$ and $n_M^G(t) - n_L^G(t)$, then $n_M^G(y) \leq n_L^G(y) \forall y > t$. Since this follows directly from the elasticity condition.

Part (ii) of proposition 3 shows that Jakobsson's condition precludes multiple-crossing of the schedules $n_i^G$. This, indeed, is how the Jakobsson condition works. From the discussion in section 2 we know that any progressivity comparison conditional on some $G \in \Lambda$ may be reduced to an equal-yield comparison between surrogate schedules $n_i^G$, the elasticities of which everywhere equal those of the underlying schedules $n_i$; the elasticity condition then ensures that, irrespective of $G$, $n_M^G$ single-crosses $n_L^G$, and Lorenz domination of $F_L|G$ by $F_M|G$ then follows from proposition 1.

It must be emphasised that while Jakobsson's condition implies single-crossing, the reverse is not true: there exist cases in which single-crossing is
satisfied for some $G$ but $\phi_M - \phi_L$ changes sign on $Y$. Consider, for instance, the replacement of a linear income tax of the form:

$$n_L(y) - ty + (1-t)y, \quad \forall y \in Y,$$

by a piecewise linear one of the form:

$$n_M(y) = \begin{cases} t_1 y + (1-t_1)y, & y \in [y, \alpha], \\ t_1 y + (t_2 - t_1)\alpha + (1-t_2)y, & y \in (\alpha, y']. \end{cases}$$

Let $G$ be such that the two taxes raise the same revenue, and suppose that $t_1 < t$. Then it is easy to check that $n_M^G$ single-crosses $n_L^G$. But it may also be verified that whilst $\phi_M(y) < \phi_L(y)$ for $y \in (\alpha, y'$), $\phi_M(y) > \phi_L(y)$ for $y \in (y, \alpha)$. The approach suggested in this paper provides a useful framework for statements on relative progressivity in such cases as this: we may say that single-crossing of $\phi_M$ by $\phi_L$ ensures that $T_M$ is more progressive than $T_L$ for $G$, although $T_M$ is not uniformly more progressive than $T_L$. In this way we can draw attention to the strong distributional implications of a switch from one tax to the other whilst emphasising that these implications are conditional upon a particular distribution of pre-tax income and are not intrinsic to the tax schedules themselves.

One other feature of this example may be noted. Jakobsson's proposition implies that the distribution of income above $\alpha$ is more equal under $T_M$ than under $T_L$ and that the reverse is true below $\alpha$. Indeed, it is clear that the Jakobsson condition is necessary and sufficient for $T_M$ to be more progressive than $T_L$ in a stronger sense than that of Lorenz domination of $F_L | G$ by $F_M | G$, namely that of Lorenz domination over each subinterval of $Y$ irrespective of $G$. Thus, unambiguous statements on the relative distributional implications of alternative taxes both locally and globally are available if and only if the taxes concerned can be ordered by uniform progressivity.

4. Concluding remarks

This paper has described the use of a single-crossing condition in comparisons between tax systems. This condition corresponds precisely to the popular notion that an increase in progressivity involves taking from the rich to give to the poor in equal-yield comparisons. It has been shown that single-crossing is closely related to the necessary and sufficient condition, derived by Jakobsson, for the distributional implications of alternative taxes to be inferred from tax schedules alone; that is, in the terminology suggested here, for taxes to be ranked by uniform progressivity.
We believe that the concepts discussed in this paper provide a useful framework for progressivity statements. Consider, for instance, the question: Is the current system of income taxation in the U.K. progressive? In terms of the standard definition of a 'progressive' tax as one whose average rate is everywhere increasing the answer is a firm 'No', for there is a narrow range of income — between the ceiling for National Insurance contributions and the lowest income on which income tax is levied at higher rates — over which the average rate falls. Nevertheless, the distribution of post-tax income Lorenz dominates that of pre-tax income, so that the distributional implications of the tax are, in practice, about as clear-cut as one can ever hope to find. Thus, although one might reasonably wish to characterize income taxation in the U.K. as progressive, it is impossible to justify doing so by referring to the standard definition alone. However, the dilemma is easily resolved by using the concepts discussed in this paper. Note first that the textbook definition of a progressive tax is precisely equivalent to the definition of such a tax as one that is uniformly more progressive than a proportional tax. Then the situation is fully described by saying that the U.K. income tax is more progressive than proportional taxation given the distribution of pre-tax income in the U.K., but is not uniformly so.

References