Piecewise linear tax functions, progressivity, and the principle of equal sacrifice

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Abstract

In this paper we discuss and criticise the conclusion asserted by Mitra and Ok (Mitra, T., Ok, E., 1996, Personal income taxation and the principle of equal sacrifice revisited, International Economic Review 37, 925–948), that when we restrict our attention to piecewise linear tax functions the principle of equal sacrifice implies progressivity. © 1999 Elsevier Science S.A. All rights reserved.

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1. Introduction

In recent years, there has been a revival of interest in the principle of equal sacrifice as a justification for progressive taxation.¹ At least starting from Samuelson (1947), who most clearly stated it, the established view among the economists has been that this principle is inconclusive, in the sense that progressivity does not necessarily follow from its application when we allow the representative agent’s utility function to satisfy standard assumptions. Mitra and Ok (1996) have recently called this conclusion into question, by showing that, given certain restrictions on the admissible utility functions, a piecewise linear tax function cannot be an equal sacrifice tax function when it is regressive; namely, they show² that when is a piecewise linear tax function, and is a utility function such that the equal sacrifice condition


²See their Theorem 3.5 and Corollary 3.6.
\[ u(x) - u(x - t(x)) = \text{constant} \quad \forall x > 0 \]  \hspace{1cm} (1)

is satisfied, it cannot be that simultaneously \( u \) is concave and differentiable near the origin (i.e. it is differentiable on an open interval \( I = (0, a) \), with \( a > 0 \) however small) and \( t \) is not convex. Mitra and Ok conclude that Samuelson’s statement is to be rejected, and equal sacrifice does imply progressivity, at least in the limited context of piecewise linear tax functions.

By restricting our attention to the simplest case of piecewise linear tax function, that of a two-brackets function, we will show that \( t \), regardless of the fact that it is progressive or regressive:

1. is always an equal sacrifice tax function with respect to a class of increasing and concave utility functions;
2. is an equal sacrifice tax function only with respect to a class of concave utility functions which are not differentiable in a (countably) infinite number of points;
3. can always be approximated by a differentiable tax function which is equal sacrifice with respect to a class of increasing, concave and everywhere differentiable utility functions.

It follows that Mitra and Ok’s conclusion is misleading: if we require that the utility function is differentiable, it does not really make a difference whether \( t \) is progressive or regressive; rather, it is piecewise linearity which is critical.

2. Results

The general form of a two-brackets piecewise linear tax function is

\[
t(x) = \begin{cases} 
  t_1 x & \text{for } x \leq \beta \\
  t_1 \beta + t_2 (x - \beta) & \text{for } x > \beta 
\end{cases}
\]  \hspace{1cm} (2)

where \( 0 \leq t_1, t_2 < 1 \); clearly, the tax is progressive (and the tax function is convex) if \( t_1 < t_2 \), regressive if conversely \( t_1 > t_2 \). At \( \beta \) the tax function is not differentiable; in order to have \( t'(x) \) defined for every \( x > 0 \), we posit that \( t'(\beta) = t_1 \).

It is useful to define the post-tax function \( f(x) = x - t(x) \), which represents the disposable income after the tax as a function of the income before the tax. It is also useful to express the equal sacrifice condition w.r.t. a concave function \( u \) in terms of one-sided first derivatives:

\[
u'(x) = u'(f(x)) f'(x) \quad \forall x > 0.
\]  \hspace{1cm} (3)

If (1) holds, then the one-sided derivative of \( u \) satisfies (3). Conversely, by integrating both sides of (3) on \( [\bar{x}, x] \) with \( \bar{x} > 0 \), we have \( u(x) - u(f(x)) = u(\bar{x}) - u(f(\bar{x})) \) for all \( x > 0 \): this means that if for all \( x > 0 \) condition \( v(x) = v(f(x)) f'(x) \) holds for some non-increasing and positive function \( v \), then a concave equal sacrifice representation \( u \) of \( t \) can be obtained simply by integrating \( v \), and \( v(x) = u'(x) \).

\(^3\)The authors themselves admit in a subsequent article, which indeed follows a very different route to tackle the problem, that their hypotheses are excessively demanding (see Mitra and Ok, 1997, p. 318).
Condition (3) constrains \( u' \) in that once we have fixed its value on \( \mathcal{B} = (\beta, f^{-1}(\beta)) \), if it satisfies (3) for a given tax function \( t \), \( u'(x) \) is fully determined on the whole \( \mathbb{R}_{++} \). In particular, when \( t(x) \) is expressed by (2), by applying recursively condition (3), we have for \( x \in f^{-n}(\mathcal{B}) \) and \( n > 0 \):

\[
u'(x) = u'(f^n(x)) \prod_{k=0}^{n-1} f'(f^k(x)) = u'(f^n(x))(1 - t_2)^n
\]

with \( f^n(x) \in \mathcal{B} \). For \( x \in f^n(\mathcal{B}) \) and \( n > 0 \) we get instead

\[
u'(x) = \frac{u'(f^{-n}(x))}{\prod_{k=0}^{n-1} f''(f^{-k}(x))} = \frac{u'(f^{-n}(x))}{(1 - t_2)(1 - t_1)^{n-1}}
\]

with \( f^{-n}(x) \in \mathcal{B} \).

As an illustrative example, consider a function which is constant and equal to \( c \) on \( \mathcal{B} \); by applying (4) and (5) recursively, we can derive its value for every \( x > 0 \):

\[
u'(x) = \begin{cases} \ldots & \text{for } f^n(\beta) < x \leq f^{n+1}(\beta) \\ \frac{c}{(1 - t_1)(1 - t_2)^{n-1}} & \text{for } f(\beta) < x \leq \beta \\ \ldots & \text{for } \beta < x \leq f^{-1}(\beta) \\ \frac{c}{1 - t_1} & \text{for } f^{-1}(\beta) < x \leq f^{-2}(\beta) \\ c & \text{for } f^{-2}(\beta) < x \leq f^{-3}(\beta) \\ \ldots & \text{for } f^{-3}(\beta) < x \leq f^{-4}(\beta) \\
\ldots & \text{for } f^{-4}(\beta) < x \leq f^{-5}(\beta) \\
\ldots & \text{for } f^{-5}(\beta) < x \leq f^{-6}(\beta) \end{cases}
\]

This function is clearly positive and nonincreasing on \( \mathbb{R}_{++} \), and by integrating it we obtain an increasing, concave, piecewise linear utility function \( u \) which is an equal sacrifice representation of the tax function (2).

Note that condition (3) puts no restriction on the behavior of \( u' \) inside the interval \( \mathcal{B} \), leaving in this respect a great degree of freedom. In general, the following will be true:

**Proposition 1.** A tax function \( t \) described by (2) is always an equal sacrifice tax function with respect to an increasing and concave utility function. Such utility function is defined by

\[u(x) = \begin{cases} c & \text{for } f^n(\beta) < x \leq f^{n+1}(\beta) \\ \frac{c}{(1 - t_1)(1 - t_2)^{n-1}} & \text{for } f(\beta) < x \leq \beta \\ \ldots & \text{for } \beta < x \leq f^{-1}(\beta) \\ \frac{c}{1 - t_1} & \text{for } f^{-1}(\beta) < x \leq f^{-2}(\beta) \\ c & \text{for } f^{-2}(\beta) < x \leq f^{-3}(\beta) \\ \ldots & \text{for } f^{-3}(\beta) < x \leq f^{-4}(\beta) \\
\ldots & \text{for } f^{-4}(\beta) < x \leq f^{-5}(\beta) \\
\ldots & \text{for } f^{-5}(\beta) < x \leq f^{-6}(\beta) \end{cases}
\]

\[\text{...}
\]

In what follows let \( f(\mathcal{B}) \) be the image of the set \( \mathcal{B} \) under the function \( f \). The index as in \( f^n \) denotes the compounded function \( f \circ f \circ \cdots \circ f \) with \( n \) iterations of \( f \); \( f^{-n} \) is the inverse of \( f^n \); \( f^n(x) = x \).

This proposition is somehow stronger than Mitra and Ok’s Theorem 2.2 in that here we have that the utility function can be concave. At the same time it differs from their Lemma 3.1 in that here no difference emerges between progressive and regressive tax functions: this is because Mitra and Ok restrict their attention to utility functions which are differentiable near the origin. See below.

\(^5\)Being irrelevant the absolute value assumed by \( u \), the lower extreme of integration could be any value where \( u' \) is defined.
\[ u(x) = \int_{\beta}^{x} u'(z) dz \] (7)

where \( u' \) is any function which: (a) satisfies (4) and (5); (b) is positive and nonincreasing in the interior of \( \mathcal{B} \); (c) is such that

\[ \frac{u'(\beta)}{(1-t_1)} \geq \lim_{x \to \beta^+} \frac{u'(x)}{(1-t_2)} \] (8)

and

\[ u'(\beta) \geq \lim_{x \to \beta^+} u'(x). \] (9)

The requirement that \( u' \) is nonincreasing in the interior of \( \mathcal{B} \), together with conditions (8) and (9), are necessary and sufficient for \( u' \) to be nonincreasing on \( \mathbb{R}_{++} \); they must be satisfied to ensure that \( u \) is concave.\(^7\) In fact, condition (8) directly follows from the requirement that

\[ u'(f^n(\beta)) \geq \lim_{x \to f^n(\beta)^+} u'(x) \] (10)

for all \( n \geq 0 \): substituting from (5) and recalling that \( f^n(\beta) \in f^{n+1}(\mathcal{B}) \) while the interval \( (f^n(\beta), f^*(\beta) + \epsilon) \) for \( \epsilon \) small is a subset of \( f^n(\mathcal{B}) \), such condition becomes

\[ \frac{u'(f^{-1}(\beta))}{(1-t_2)(1-t_1)^n} \geq \lim_{x \to f^n(\beta)^+} \frac{u'(x)}{(1-t_2)(1-t_1)^{n-1}} \] (11)

from which (8) follows. Condition (9) similarly follows from the requirement that \( u' \) does not increase at points \( \{f^{-n}(\beta)\}_{n=1}^{\infty} \).

Note that the two inequalities (8) and (9) are consistent with the requirement that \( u' \) is nonincreasing on \( \mathcal{B} \), which implies that \( \lim_{x \to \beta^+} u'(x) = u'(f^{-1}(\beta)) = u'(\beta)(1-t_2) \). Hence, the conditions that must be imposed on \( u' \) are not reciprocally incompatible.

We have not yet mentioned differentiability: indeed, \( u' \) cannot be at the same time continuous and nonincreasing on \( \mathbb{R}_{++} \). We have seen that a nonincreasing \( u' \) should satisfy (8) and (9), but this implies that when the tax is regressive \( (t_2 < t_1) \) we have

\[ \lim_{x \to f^n(\beta)^+} u'(x) = \lim_{x \to \beta^+} \frac{u'(x)}{(1-t_2)(1-t_1)^{n-1}} \leq \frac{u'(\beta)}{(1-t_2)(1-t_1)^n} < \frac{u'(\beta)}{(1-t_1)^n} = u'(f^n(\beta)). \] (12)

The function \( u' \) is discontinuous at \( f^n(\beta) \) for any \( n \geq 0 \); it follows that for any interval \( I = (0, \alpha) \) there will be an integer \( N \) such that \( f^N(\beta) \in I \). This is just Mitra and Ok’s result: when \( t \) is regressive,

\(^7\)Both conditions (8) and (9) must be valid for \( u' \) to be decreasing on \( \mathbb{R}_{++} \), though one of them is obviously redundant. Which is redundant depends on the relative values of \( t_1 \) and \( t_2 \), that is to say on the progressivity or regressivity of the tax function. With \( t_1 > t_2 \), only (9) is relevant, while with \( t_1 < t_2 \) the only relevant condition is (8).
it cannot be an equal sacrifice function with respect to a concave \( u \) whose derivative is continuous near the origin.

We observe that a similar problem is faced with progressivity, as well. In this case \( 1 - t_2 < 1 - t_1 \) so that, for \( n = 0, 1, 2 \ldots \) we have, as a consequence of (8)

\[
\lim_{x \to f^{-n}(\beta)^+} u'(x) = \lim_{x \to \beta^+} u'(x)(1 - t_2)^n \leq u'(\beta) \frac{(1 - t_2)^{n+1}}{(1 - t_1)^n} < u'(\beta)(1 - t_1)^n = u'(f^{-n}(\beta))
\]

therefore \( u' \) cannot be simultaneously continuous and nonincreasing, and \( u' \) will be discontinuous at \( f^{-n}(\beta) \) for any \( n \geq 0 \).

Our observation can be summarized in

**Proposition 2.** If \( t \) is a piecewise linear tax function defined by (2) with \( t_1 \neq t_2 \), and it is an equal sacrifice tax function with respect to a concave \( u \), then there are (countably) infinite points where \( u \) is not differentiable.

The existence of a symmetry between progressivity and regressivity is stressed here, a point which does not emerge in Mitra and Ok’s contribution. In both cases, piecewise linearity gives place to an infinite number of points where \( u' \) is discontinuous; the difference is that these points are in the interval \( (0, \beta] \) when \( t \) is regressive, while they are in the interval \( [\beta, +\infty) \) with progressivity. Therefore, Mitra and Ok’s restriction about differentiability near the origin appears to be quite arbitrary: excluding only those utility functions whose kinks are concentrated near the origin predetermines the result against regressivity.\(^8\)

We now turn to the role played by the assumption of piecewise linearity. To illustrate it, we allow a slight modification of the tax function \( t \): we introduce \( \tilde{t} \), which is everywhere differentiable and such that \( |t(x) - \tilde{t}(x)| < \delta \) with \( \delta > 0 \) small.

Take for example the tax function \( \tilde{t}(x) \equiv t(x) + \tau(x) \) such that

\[
\tau(x) \text{ is increasing on } (\beta, \beta + \epsilon);
\]

\[
\tau(x) = 0 \text{ for } x = \beta;
\]

\[
\tau(x) = k \text{ for } x = \beta + \epsilon;
\]

\[
\lim_{x \to \beta^+} \tau'(x) = t_1 - t_2 \text{ and } \tau'(\beta + \epsilon) = 0
\]

\[
\tau'(x) \text{ is continuous and decreasing on } (\beta, \beta + \epsilon).
\]

Note that the derivative of \( \tilde{t} \) is continuous for \( x > 0 \). Moreover, \( |t(x) - \tilde{t}(x)| = \tau(x) \): \( \tilde{t} \) is an approximation of \( t \) if \( k \) is sufficiently close to zero; clearly, the smaller is \( k \), the ‘steeper’ is \( \tau'(x) \) on \( (\beta, \beta + \epsilon) \).

\(^8\)Mitra and Ok defend this assumption by asserting that ‘it seems extremely difficult to argue that individuals’ utility functions for income are concave and strictly increasing functions which are not differentiable in any neighborhood of the origin’. But, in the light of Proposition 2, it is not clear why we must prefer that the sequence of points in which \( u \) is not differentiable should be diverging to infinity rather then converging to the origin (note that the utility function is not defined at the origin).
Recall that \( f(x) \) has now the following expression
\[
\begin{align*}
  f(x) &= \begin{cases} 
    (1 - t_1)x & \text{for } x \leq \beta \\
    (1 - t_2)x - \tau(x) - (t_1 - t_2)\beta & \text{for } \beta < x < \beta + \epsilon \\
    (1 - t_2)x - k - (t_1 - t_2)\beta & \text{for } x \geq \beta + \epsilon ;
  \end{cases}
\end{align*}
\] (14)
it is still a continuous and increasing function — and so it is invertible.

If we follow the same procedure as above to build \( u \), we find that concavity and equal sacrifice are not any more incompatible with differentiability, even for a regressive tax function. We can state the following:9

**Proposition 3.** A tax function \( \tilde{t}(x) = t(x) + \tau(x) \), where \( t \) is described by (2) and \( \tau \) is defined by the conditions \( \tau_1 - \tau_5 \), is an equal sacrifice tax function with respect to a utility function which is increasing, concave and differentiable on \( \mathbb{R}_{++} \). Such utility function is defined by
\[
  u(x) = \int_{\beta}^{x} u'(z)dz
\] (15)
where \( u' \) is any function which is continuous, nonincreasing, strictly positive on \( \mathbb{R} \), satisfies
\[
  \lim_{x \to \beta^+} u'(x) = u'(\beta) = \frac{u'(f^{-1}(\beta))}{1 - t_2}
\] (16)
and whose value on \( \mathbb{R}_{++} \setminus \beta \) is obtained by applying (3) recursively.

It follows from this proposition that, even when we do require (as Mitra and Ok do) that the utility function is differentiable near the origin, the conclusion that equal sacrifice rules out regressive tax functions is not robust with regard to a slight departure from the assumption that the tax function is not differentiable.

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**References**


9The proof is available upon request.