# A Possibility Theorem on Majority Decisions

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# A POSSIBILITY THEOREM ON MAJORITY DECISIONS

BY AMARTYA K. SEN

In this note we show the consistency of majority decisions under preference conditions that are more general than Single-Peaked Preferences (Arrow [1], Black [2]), Single-Caved Preferences (Inada [5]), Preferences separable into Two Groups (Inada [5]), and Latin-Square-less Preferences (Ward [12]). In the first part of the note, the underlying concepts and approach are introduced; in the second part the theorem is stated and proved; and in the third part its relationship with other sufficiency conditions is discussed.

## 1

FOR ANY set of alternatives, the voters who are indifferent between all of them raise some peculiar problems for the consistency of majority decisions. So we first separate them out.

DEFINITION: A *Concerned Individual* with respect to a set of alternatives is one who is not indifferent between all the alternatives. An individual who is indifferent between all alternatives is *Unconcerned.*

Each individual is assumed to have a weak ordering ranging over all alternatives. Notationally, $x\ R_i\ y$ stands for the $i$th individual regarding alternative $x$ to be at least as good as alternative $y$, and $x\ P_i\ y$ stands for the $i$th individual preferring $x$ to $y$, which is equivalent to $x\ R_i\ y$, and *not* $y\ R_i\ x$. We follow Arrow (rather than Black) in defining the method of majority decisions. Let $x\ R\ y$ stand for $x$ being socially regarded at least as good as $y$, and $x\ P\ y$ stands for the corresponding social preference, i.e., for $x\ R\ y$, and *not* $y\ R\ x$.[1]

DEFINITION: *The method of majority decisions* means that $x\ R\ y$ if and only if the number of individuals such that $x\ R_i\ y$ is at least as great as the number of individuals such that $y\ R_i\ x$.[2]

The consistency of the majority decisions for a set of two alternatives has been shown by Arrow, but inconsistency can arise when there are three or more alternatives. We shall find the following lemma useful.[3]

[1] We shall be avoiding discussion of the important problems arising from "strategic masking or distorting of preferences" (see Rothenberg [9]; see also Majumdar [7] and Luce and Raiffa [6]). Also the lack of a one-to-one correspondence between preferences and voting in a model of continuous utility maximization, no matter whether the utility from voting is taken to be zero, positive, or negative (see Sen [10]).

[2] It is easy to check that the method of majority decisions satisfies the conditions of *reflexivity* ($x\ R\ x$), and *connectedness* (either $x\ R\ y$ or $y\ R\ x$). The only property of a weak social ordering that is in doubt is *transitivity*.

[3] Ward [12] discusses this, and though he confines his attention to a strict social ordering corresponding to the $P$-relation, which does not have the property of reflexivity and connectedness, our proof is nevertheless very similar to his.

LEMMA 1: *If among a set of alternatives there is no triple such that, given the set of individual preferences, the method of majority decision gives intransitive results between them, then the method will give consistent results for the entire set of alternatives.*

PROOF: Suppose, to the contrary, there is no such triple, but there is nevertheless an inconsistency involving more than three (say, $n$) alternatives. We have, let us say, the following inconsistent result:

$$A_1\, R\, A_2,\ A_2\, R\, A_3,\ \ldots,\ A_{m-1}\, R\, A_m,\ A_m\, P\, A_{m+1},$$
$$A_{m+1}\, R\, A_{m+2},\ \ldots,\ A_{n-1}\, R\, A_n,\ A_n\, R\, A_1\,.$$

Take the triple $(A_m, A_{m+1}, A_{m+2})$, which like all other triples must yield consistent results. Hence, $A_m\, P\, A_{m+2}$. Proceeding this way, we get $A_m\, P\, A_1$. Now, take the triple $(A_{m-1}, A_m, A_1)$, which, giving consistent results, yields $A_{m-1}\, P\, A_1$. Proceeding this way, we have $A_3\, P\, A_1$. But since we have $A_1\, R\, A_2$ and $A_2\, R\, A_3$, we also have $A_1\, R\, A_3$. This is a contradiction.

Similar contradictions occur, *a fortiori*, when there is more than one $P$ (preference) relation in the series. This proves that if in a set of alternatives there is no triple with intransitivity, majority decisions for the whole set must be consistent.

An alternative, $x$, can be called "best" (strictly, "among the best") of three alternatives ($x$, $y$, and $z$), for a given individual $i$, if and only if, $x\, R_i\, y$, and $x\, R_i\, z$. Similarly, $x$ is a "worst" alternative for him, if and only if $y\, R_i\, x$, and $z\, R_i\, x$. A "medium" alternative for him is defined by $x$ satisfying either the conditions that $z\, R_i\, x$ and $x\, R_i\, y$, or the conditions that $y\, R_i\, x$ and $x\, R_i\, z$.

DEFINITION: The *value* of an alternative in a triple for an individual preference ordering is its characteristic of being *"best," "worst,"* or *"medium."*

Of course, in orderings involving indifference, an alternative can have more than one value; in fact, if individuals are not "concerned," then each alternative has each value.

We now introduce the crucial definition of a Value-Restricted preference pattern. One alternative in a triple is excluded from having any one of the three values.

ASSUMPTION OF VALUE-RESTRICTED PREFERENCES: *A set of individual preferences over a triple of alternatives such that there exist one alternative and one value with the characteristic that the alternative never has that value in any individual's preference ordering, is called a Value-Restricted Preference pattern over that triple for those individuals.*

Finally, we introduce some notation. The number of individuals for whom $x\, R_i\, y$ is referred to by $N(x \geqq y)$; the number for whom $x\, P_i\, y$ is called $N(x > y)$. The number for whom $x\, P_i\, y$ and $z\, P_i\, y$ is referred to as $N(x > y,\ y < z)$. The number who hold $x\, R_i\, y$ and $y\, R_i\, z$ is referred to as $N(x \geqq y \geqq z)$. These examples illustrate the principles of the notation, and the same system is employed for corresponding concepts.

## 2

The theorem to be proved is stated in terms of Arrow's Social Welfare Function, which clarifies the relevance of our result.

THEOREM 1 (POSSIBILITY THEOREM FOR VALUE-RESTRICTED PREFERENCES): *The method of majority decision is a social welfare function satisfying Arrow's Conditions 2–5 and the consistency condition for any number of alternatives, provided the preferences of concerned individuals over every triple of alternatives is Value-Restricted, and the number of concerned individuals for every triple is odd.*

All the stated conditions, except the requirement of transitivity, have been shown by Arrow [**1**] to hold for the method of majority decision.[4] So we need be concerned only with transitivity. Thanks to Lemma 1, we only have to make sure that transitivity holds for all triples of alternatives.

It is easy to check that with three alternatives, $x$, $y$, and $z$, all intransitivities must imply one and exactly one of the following, which for convenience we give two (rather arbitrary) names:

*Forward Circle*: $x\ R\ y$, $y\ R\ z$, and $z\ R\ x$.

*Backward Circle*: $y\ R\ x$, $x\ R\ z$, and $z\ R\ y$.

The existence of one of these circles is *necessary* for intransitivity, but it is not sufficient, for which we need, in addition, the absence of the other.

First take the forward circle. Since $x\ R\ y$ and $y\ R\ z$,

$$N(x \geqq y) \geqq N(y \geqq x), \text{ and } N(y \geqq z) \geqq N(z \geqq y).$$

By adding the two inequalities, we have

$$(1) \qquad N(x \geqq y) + N(y \geqq z) \geqq N(z \geqq y) + N(y \geqq x).$$

Now the left-hand side equals

$$\begin{aligned} \text{LHS} &= N(x \geqq y \geqq z) + N(x \geqq y < z) + N(y \geqq z) \\ &= N(x \geqq y \geqq z) + [N - N(x < y < z)], \end{aligned}$$

where $N$ is the total number of individuals. Similarly, the right-hand side equals

$$\text{RHS} = N(z \geqq y \geqq x) + [N - N(z < y < x)].$$

After simplifying, the inequality (1) becomes:

$$(1.1) \qquad N(x \geqq y \geqq z) + N(x > y > z) \geqq N(z \geqq y \geqq x) + N(z > y > x).$$

Similarly, taking respectively the combination $y\ R\ z$ and $z\ R\ x$, and the combination

[4] See also May [**8**] showing that for pair-wise choice, the simple majority rule is the only rule satisfying the properties of "decisiveness," "anonymity," "neutrality," and "positive responsiveness."

$z\ R\ x$ and $x\ R\ y$, we obtain:

(1.2) $N(y \geqq z \geqq x) + N(y > z > x) \geqq N(x \geqq z \geqq y) + N(x > z > y)$,

(1.3) $N(z \geqq x \geqq y) + N(z > x > y) \geqq N(y \geqq x \geqq z) + N(y > x > z)$.

The fulfillment of (1.1), (1.2), and (1.3) simultaneously is a necessary condition for the forward circle (though not a sufficient condition).

Consider now the special case when the only people who hold both $x\ R_i\ y$ and $y\ R_i\ z$ are the ones who are indifferent between all three (i.e., are not "concerned" about these three). We then have:

$N(x \geqq y \geqq z) = N(x = y = z)$ and $N(x > y > z) = 0$, so that the left-hand side of (1.1) becomes

$\text{LHS} = N(x \geqq y \geqq z) + N(x > y > z) = N(x = y = z)$.

Since $N(z \geqq y \geqq x)$ must be at least as great as $N(x = y = z)$, the RHS too must equal the latter so that (if intransitivity is to take place), no "concerned individual" should hold $z\ R_i\ y$ and $y\ R_i\ x$ together.

Under these conditions, all "concerned individuals" can only have strict preference between $x$ and $y$, and $y$ and $z$, and these two preferences must run in opposite directions. So the total number of individuals, $N$, can be strictly partitioned into three groups:

$$N = N(x = y = z) + N(x > y,\ y < z) + N(x < y,\ y > z).$$

Now since we have $x\ R\ y$ and $y\ R\ z$, we require, respectively, the following two conditions:

$$N(x > y,\ y < z) \geqq N(x < y,\ y > z),$$

$$N(x < y,\ y > z) \geqq N(x > y,\ y < z).$$

This means that the two sides are exactly equal. But since the two sides add up to the total number of concerned individuals, this can happen only if the number of such individuals is even. But by assumption of the theorem, this number is odd. Thus, there is a contradiction.

It is thus established that a forward circle is impossible if:

(i) $N(x \geqq y \geqq z) = N(x = y = z)$.

Identically, it can be shown that such a circle is impossible, if either of the two following conditions hold:

(ii) $N(y \geqq z \geqq x) = N(x = y = z)$,

(iii) $N(z \geqq x \geqq y) = N(x = y = z)$.

An exactly similar procedure establishes that a backward circle is impossible if any one of the following three conditions hold:

(iv) $N(y \geqq x \geqq z) = N(x = y = z)$,

(v) $N(x \geqq z \geqq y) = N(x = y = z)$,

(vi) $N(z \geqq y \geqq x) = N(x = y = z)$.

We can glorify this result into a lemma which we use for the proof of our theorem.

LEMMA 2: *If for any triple of alternatives* $(x, y, z)$, *at least one of the conditions* (i)-(iii) *and at least one of the conditions* (iv)-(vi) *hold and if the number of concerned individuals for the triple is odd, then the triple must yield transitive majority decisions.*

PROOF OF THEOREM I: Since the preference ordering of every concerned individual is Value-Restricted for every triple, in each triple we can identify an alternative (say, $x$) such that it cannot have a given value (say, "best"). If $x$ cannot be best for concerned individuals, then people satisfying respectively $(x\, R_i\, y,\ y\, R_i\, z)$ or $(x\, R_i\, z,\ z\, R_i\, y)$ must be unconcerned. Hence conditions (i) and (v) hold. Similarly if $y$ or $z$ cannot be best in any concerned individual's preferences, we have, respectively (ii) and (iv), and (iii) and (vi).

When $x$, $y$, or $z$ cannot be "worst" for any concerned individual, we have respectively, (ii) and (vi), (iii) and (v), and (i) and (iv). When $x$, $y$, or $z$ cannot be "medium" for any concerned individual, we have respectively, (iii) and (iv), (i) and (vi), and (ii) and (v).

Thus, in every case of Value-Restricted Preferences, at least one of the conditions (i)–(iii) and at least one of the conditions (iv)–(vi) are satisfied. Since the number of concerned individuals is odd for every triple, it follows from Lemma 2 that majority decisions are transitive for every triple.

The theorem now follows from Lemma 1. *Q.E.D.*

## 3

We can now compare the theorem proved here with some of the ones that have preceded it. The most well-known sufficiency condition is Black's [2] and Arrow's [1] assumption of Single-Peaked Preferences.[5] Inada [5] has pointed out recently that Arrow assumes Single-Peaked Preferences for *all* alternatives, but in his proof uses that only for *all* triples, which is a weaker condition. The assumption of single-peakedness means that there is a strong ordering $S$ for arranging the alternatives such that if $y$ is "between" $x$ and $z$, $x\, R_i\, y \rightarrow y\, P_i\, z$. This means that Single-Peaked Preferences over a triple imply: $Not(x\, R_i\, y$, and $z\, R_i\, y)$, i.e., $y$ cannot be given the value "worst." The other two cases of Single-Peakedness

[5] There are slight differences between the two versions, and indeed between Black's original presentation [2], which rules out indifference, and his later presentation [3], which even allows individuals to be indifferent between all alternatives, not allowed by Arrow. Since, however, Black assumes that the number of all individuals is odd and not (as we do) that the number of *concerned individuals* is odd, his theorem is, strictly speaking, incorrect. See the example on the top of page 39 of Dummett and Farquharson [4], which is Single-Peaked in Black's sense [3], but shows intransitivity.

imply that $x$ is not a "worst" alternative, and that $z$ is not a "worst" alternative. The theorem proved in this note covers all the cases.[6]

Inada's Possibility Theorem for Single-Caved Preferences [5] uses a concept opposite to that of Arrow's Single-Peakedness.[7] If $y$ is "between" $x$ and $z$, then $y R_i x \rightarrow z P_i y$. This means: $Not(y R_i x$, and $y R_i z)$. This only requires that $y$ not have the value "best." The other cases of Single-Caved Preferences rule out, respectively, $x$ being "best," and $z$ being "best." All these cases are, therefore, subsumed in the theorem proved in the last section.

We can now consider Inada's theorem for preferences such that the "set of alternatives [are] *separable into two groups*" A and B, with any alternative in A being always preferred to any alternative in B, or vice versa. Such separability for *all* alternatives is equivalent to the separability for all triples.[8] Suppose A consists of $x$ and B of $(y, z)$. Now, we have: (1) $x P_i y$ if and only if $x P_i z$, (2) $y P_i x$ if and only if $z P_i x$, and (3) either $x P_i y$ or $y P_i x$. This excludes all possibilities where $x$ can be "medium." The other cases of separability into two groups exclude respectively $y$ and $z$ being "medium." All these cases are, therefore, also covered by our theorem.

Ward's [12] Latin-Square-lessness applies to cases with strong ordering, but in those cases they are exactly equivalent to Value-Restricted Preferences. Under these special conditions, Ward shows that if among the preference orderings of all individuals for any triple, there are not three that form a latin square, then majority decisions (in the sense of strict orderings, $P$-relation) will be transitive.[9] Assuming strong ordering with our Value-Restricted preferences, it is easy to check that a latin square cannot be formed of the permitted preferences of the concerned individuals.[10] Thus Ward's cases are subsumed in the cases covered here, and hold for majority decisions interpreted both as the $R$-relation and the $P$-relation.

It should be added that the theorems which are shown to be subsumed by the

[6] The extension proposed by Dummett and Farquharson [4] is, however, not covered. But they solve a different problem from ours, viz., when will a set of alternatives "have a top," i.e., "there should be at least one outcome $x$ such that for every other outcome $y$ some majority regards $x$ as at least as good as $y$" (p. 40). Dummett and Farquharson show that for this it is sufficient to assume that "of every three outcomes there is one which no voter thinks *worse* than *both* the other two" (p. 40). It is easily checked, however, that this is not sufficient for the requirement of transitivity of majority decisions. Take, for example, the following three orderings: (1) $x P_1 y P_1 z$; (2) $y P_2 x I_2 z$; (3) $z P_3 x P_3 y$. This satisfies Dummet and Farquharson's restriction ($x$ is not strictly worse than $y$ and $z$ for any one), and the alternatives "have a top," viz., $x$. But there is intransitivity, since $x P y$, $y P z$, but $z I x$.

[7] Cf. Vickrey [11, pp. 514–5]. Vickrey calls it "single-troughed preferences" and confines his attention to strong orderings only.

[8] See Inada [5, pp. 530–1].

[9] See also Vickrey [11, pp. 513–6], who discusses cases similar to Ward's [12].

[10] Value-Restricted Preferences can be viewed as a weak-ordering generalization of Latin-Squarelessness. We can define Value-Restricted Preferences as preferences such that a latin square cannot be formed out of them by *any permutation of its indifferent alternatives*.

one developed here all assume that (a) the number of *all* individuals is odd, and (b) there are *no* individuals who are indifferent between all three alternatives in any triple. We have not assumed (b), and have assumed instead of (a) that the number of *concerned* individuals is odd. When (a) and (b) are fulfilled, our condition is also fulfilled, but there are cases when our condition is fulfilled but (a) and (b) are not. For example, when there are an even number of individuals, an odd number of whom are not concerned in any triple; this violates both (a) and (b). Thus, in this respect also, there is a slight generalization.

When this question of unconcerned individuals is ignored, however, and $N(x=y=z)$ is taken to be zero, the set of our cases partitions exactly into three proper subsets of cases: (I) When one alternative cannot be "best," equivalent to Single-Caved Preferences; (II) When one alternative cannot be "worst," equivalent to Single-Peaked Preferences, and (III) when one alternative cannot be "medium," equivalent to the case of alternatives Separable into Two Groups.

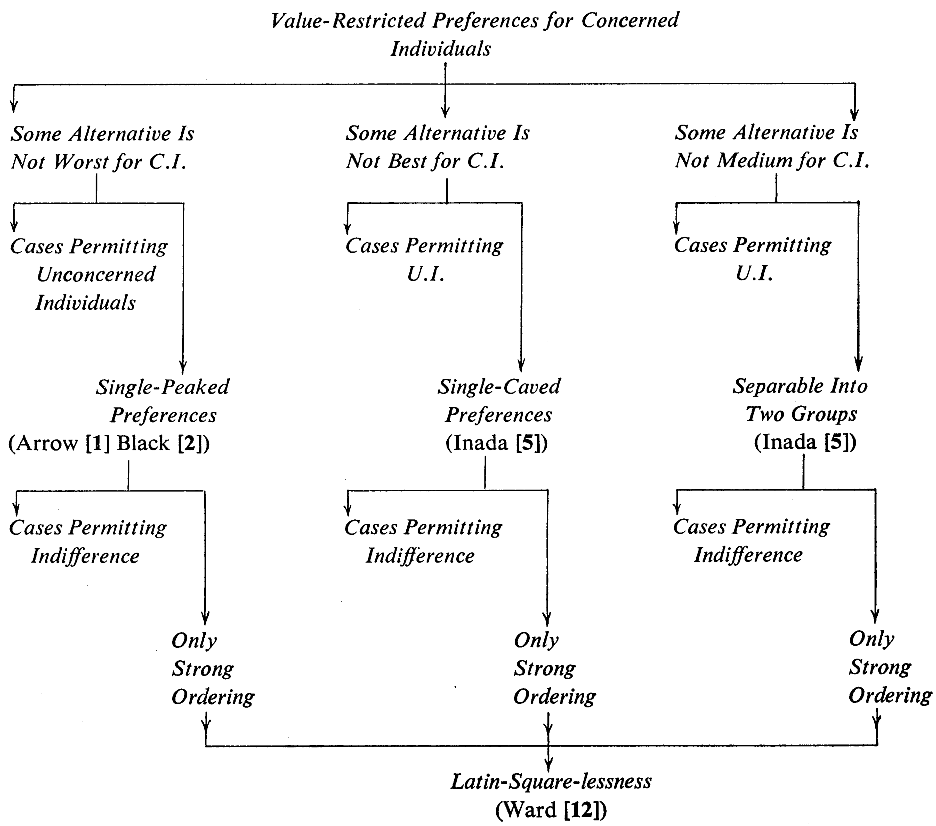


FIGURE 1.—Restriction on preferences for each triple.

Since mine is only an attempt to extend the pioneering work of Arrow, Black, Inada, Vickrey, and Ward, I end with a chart explaining the relationship between the assumptions made in the theorem proved here and the assumptions made in their respective theorems. (See figure 1.)

It is important to note, however, that we need not assume that the same "restriction" holds for one triple that holds for another. If it is possible for one triple to have, say, Single-Peaked Preferences, and another triple to have, say, Single-Caved Preferences, and so on, the Theorem will still be valid, though such cases will not be covered by either of the three separate theorems involving Single-Peakedness, Single-Cavedness, and Separability. Consider, for example, the following set of five preference orderings over four alternatives $(w, x, y, z)$:

1. $w\, I_1\, x\, P_1\, y\, P_1\, z$ ,
2. $x\, I_2\, w\, P_2\, z\, P_2\, y$ ,
3. $z\, I_3\, x\, P_3\, y\, P_3\, w$ ,
4. $z\, P_4\, y\, I_4\, x\, P_4\, w$ ,
5. $z\, P_5\, y\, P_5\, x\, P_5\, w$ .

All the four possible triples are Value-Restricted: $(w, x, y)$ is Single-Peaked ($x$ is not worst); $(x, y, z)$ is Single-Caved ($y$ is not best); $(w, x, z)$ is Single-Peaked ($x$ is not worst); $(w, y, z)$ is both Separable into Two Groups ($w$ is not medium) and Single-Caved ($y$ is not best). The majority decisions are all transitive, yielding the social ordering: $x\, I\, z\, P\, y\, P\, w$. This consistency is covered by Theorem I, though it is not covered by any of the individual theorems of Single-peakedness (Arrow [**1**]), Single-Cavedness (Inada [**5**]), Separation into Two Groups (Inada [**5**]), and Latin-Square-lessness (Ward [**12**]).

It would seem that Value-Restricted Preferences will cover a variety of practical cases. A comparatively limited measure of agreement seems to be sufficient to guarantee consistent majority decisions, and to get from it a Social Welfare Function with the other properties specified by Arrow.

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## REFERENCES

[1] Arrow, K. J.: *Social Choice and Individual Values*, New York: Wiley, 1951, Chapter VII.
[2] Black, D.: "On the Rationale of Group Decision-Making," *Journal of Political Economy*, 56 (February, 1948).
[3] ———: *The Theory of Committees and Elections*, Cambridge: Cambridge University Press, 1958.
[4] Dummet, M., and R. Farquharson: "Stability in Voting," *Econometrica*, 29 (January, 1961).
[5] Inada, K.: "A Note on the Simple Majority Decision Rule," *Econometrica*, 32 (October, 1964).

[6] LUCE, R. D., AND H. RAIFFA: *Games and Decisions*, New York: Wiley, 1958, Chapter 14.
[7] MAJUMDAR, T.: "Choice and Revealed Preference," *Econometrica*, 26 (1956).
[8] MAY, K. O.: "A Set of Independent, Necessary and Sufficient Conditions for Simple Majority Decision," *Econometrica*, 20 (1952).
[9] ROTHENBERG, J.: *The Measurement of Social Welfare*, Englewood Cliffs, N.J.: Prentice-Hall, 1961, Chapter 11.
[10] SEN, A. K.: "Preferences, Votes and the Transitivity of Majority Decisions," *Review of Economic Studies*, 31 (April, 1964).
[11] VICKREY, W.: "Utility, Strategy, and Social Decision Rules," *Quarterly Journal of Economics*, 74 (November, 1960).
[12] WARD, B.: "Majority Voting and Alternative Forms of Public Enterprises," in J. Margolis (ed.), *2nd Conference in Urban Expenditure* (forthcoming).