On evaluating social welfare by sequential generalized Lorenz dominance

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Abstract

The sequential generalized Lorenz (SGL) ordering is introduced by Atkinson and Bourguignon (1987) to rank heterogeneous income distributions. It is well-known that this ordering has a strong utilitarian support. In this note, we show that one does not have to be a utilitarian to accept this welfare ordering: the SGL ordering is supported by all increasing SWFs which record an increase in overall welfare when a (cardinal) welfare transfer is made from less needy to needier.

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1. Introduction

Ranking income distributions by means of the generalized Lorenz (GL) ordering is a commonly used procedure in welfare economics. This is due to the fact that one income distribution GL dominates another if, and only if, the former yields higher total welfare than the latter in terms of all increasing and concave social utility functions and all increasing, symmetric and quasi-concave social welfare functions (SWFs).\textsuperscript{1} However, it is now well recognized that this approach to comparing income distributions is rather oversimplified, for it ignores the non-income characteristics of individuals (like marital status, family size, etc.), or for short, the 'needs'.

To provide a remedy for this, Atkinson and Bourguignon (1987) have introduced the sequential generalized Lorenz (SGL) ordering, extending the GL ordering to the more realistic case where the population is partitioned into subgroups on the basis of needs. For the SGL ordering, one income distribution dominates another if, and only if, it yields higher welfare for all admissible social utility

\textsuperscript{1}See Atkinson (1970), Dasgupta et al. (1973), Rothschild and Stiglitz (1973), Kolm (1976) and Shorrocks (1983) inter alia.

The term 'generalized Lorenz dominance' is due to Shorrocks.

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functions (satisfying certain reasonable conditions defined below) and for the utilitarian SWF. The importance of this result is indisputable, for it shows that the SGL ordering commands a considerable utilitarian support. However, it is not readily clear to what extent the ethical support of this ordering is rooted solely in utilitarianism, i.e., if the Atkinson-Bourguignon analysis can be extended beyond the realm of the utilitarian framework. In this note we shall outline one such extension.

When ranking income distributions of heterogeneous populations, there are two stages in which ethical considerations play a part. Given an income distribution and an admissible utility profile, the first stage is the aggregation of individual utility levels to obtain the welfare level of a given subpopulation; and the second stage is, in turn, the aggregation of the welfare levels of the subgroups in order to derive the overall welfare of the population. Since Atkinson and Bourguignon (1987) employed the utilitarian methodology in both of these stages, there are two possible ways of studying the command of the SGL ordering outside the boundaries of utilitarianism. In this note, we shall study only one of these possible extensions, and follow Atkinson and Bourguignon (1987) in using the utilitarian method of aggregation in the first stage. In other words, we assume that the welfare level of a subgroup is determined as the sum of individual utilities. Our main inquiry here regards the relation between the SGL ordering and the non-utilitarian SWFs that aggregate subpopulation welfare levels.

To make things clear, let \( \mathcal{W} \) stand for the largest class of SWFs (that map the distributions of subgroup welfare levels to overall welfare) such that one heterogeneous income distribution SGL dominates another if, and only if, the former yields higher social welfare in terms of all admissible utility profiles and all SWFs belonging to \( \mathcal{W} \). The main question of the present note is: What does \( \mathcal{W} \) look like?

Of course, \( \mathcal{W} \) is not empty; the utilitarian SWF belongs to \( \mathcal{W} \). One immediate conjecture is that the SGL ordering has the ethical support of GL dominance; that is, \( \mathcal{W} \) is composed of all increasing, symmetric and quasi-concave SWFs. If this were true, the ethical significance of the SGL ordering would be quite strong. Unfortunately, it turns out that this is not the case; for instance, the Rawlsian SWF does not belong to \( \mathcal{W} \). Yet, \( \mathcal{W} \) is nevertheless quite a rich class. Our main result demonstrates that it is composed of precisely those SWFs that record an increase in overall welfare whenever a (cardinal) welfare transfer is made from the less needy to the needier. We refer to such SWFs as needs-based, and think of them as SWFs that attach more welfare weight to the level of well-being of the needier. With this terminology, we may restate our main result as follows: a heterogeneous income distribution SGL dominates another if, and only if, the former yields higher social welfare in terms of all (and only) increasing and needs-based SWFs for any given admissible utility profile. This result characterizes the ethical properties of the SGL ordering with regard to the aggregation of the well-being levels of the constituent subgroups.

2. Sequential generalized Lorenz ordering

We assume that there are \( H \geq 1 \) distinct household types. For example, if \( H = 3 \), the first, the

\footnote{For illuminating reflections on this result, see Bourguignon (1989), Coulter et al. (1992), Lambert (1993), pp. 82–86, Jenkins and Lambert (1993), and Moyes (1994). Ebert (1995) approaches to the same problem by an alternative axiomatic approach.}

\footnote{We do not, of course, pretend that this assumption is without problems. Indeed, the implications of dropping the utilitarian aggregation at this stage as well constitutes an important topic for future research.}
second and the third subpopulations can be thought of as single persons, married couples and families with children, respectively. The \( i \)th category is identified by a (cardinal) utility function \( u_i \), \( i = 1, \ldots, H \). Letting \( U \) denote the class of all twice differentiable, strictly increasing and concave utility functions, we define the class of admissible (social) utility profiles as

\[ \mathcal{U}_H = \{ u = (u_1, \ldots, u_H) \in U^H : u'_i \succeq \cdots \succeq u'_H \text{ and } u''_i \succeq \cdots \succeq u''_H \} \]

The first derivative condition means that, at every income level, people in the \( i \)th group have a higher social marginal valuation of income (hence, are needier) than in the \((i + 1)\)th group. The second derivative condition states that the differences in social marginal evaluations between subgroups are decreasing in income. As noted in Atkinson and Bourguignon (1987), p. 360, “it may not be unreasonable to suppose that we become less concerned about differences in needs at higher incomes.”

We take the class of admissible income distributions for the \( i \)th category as

\[ D(\alpha) = \left\{ h \in L^1_+ [0, \alpha] : \int_0^\alpha h(t) \, dt = 1 \right\} \]

where \( L^1_+ [0, \alpha] \) is the space of nonnegative integrable functions on \([0, \alpha]\), for some \( \alpha > 0 \). Here \( \alpha \) represents an upper bound for the range of possible income levels. Without loss of generality, we shall assume that \( \alpha > 1 \).

Throughout this note, we shall follow Atkinson and Bourguignon (1987) in assuming that the marginal distribution of needs is fixed. To this end, take any income distribution in \( D(\alpha)^H \) such that the population share of the \( i \)th subgroup, denoted \( p_i \), is nonzero. Define next the class of all admissible income distributions as the set of all members of \( D(\alpha)^H \) such that the population share of the \( i \)th subgroup is \( p_i \). We denote this class (which is parametric over the chosen \( p_i \)s) by \( D_H(\alpha) \).

**Definition 2.1.** Let \( f, g \in D_H(\alpha) \). \( f = (f_1, \ldots, f_H) \) is said to sequentially generalized Lorenz (SGL) dominate \( g = (g_1, \ldots, g_H) \), denoted \( f \succ_{SGL} g \), if

\[ \sum_{i=1}^s p_i \int_0^x f_i(t) \, dt \, dx \leq \sum_{i=1}^s p_i \int_0^x g_i(t) \, dt \, dx, \quad s = 1, \ldots, H \]

for all \( y \in [0, \alpha] \). The asymmetric part of the sequential generalized Lorenz ordering, \( >_{SGL} \), is defined as usual.

**Remark 2.2.** For an homogeneous population, the SGL ordering reduces to the second-order stochastic dominance relation, known in the inequality literature as the GL ordering a lá Shorrocks (1983). It is well-known that, for any \( f, g \in D_1(\alpha) \), we have \( f \succ_{GL} g \) if, and only if, \( \int_0^\alpha v(x)f(x) \, dx \geq \int_0^\alpha v(x)g(x) \, dx \) for all \( v \in U \).

\[ ^4 \text{These conditions entail interpersonal unit comparability (cf. Sen, 1970, p. 106).} \]
Define

\[ I(u_i, f_i) = p_i \int_0^\alpha u_i(t) f_i(t) \, dt, \quad i = 1, \ldots, H \]

where \( u_i \in U \) and \( f_i \in D(\alpha) \). If \( f_i \in D(\alpha) \) is the income distribution of subgroup \( i \), then \( I(u_i, f_i) \) is the total welfare level of subgroup \( i \) according to the utilitarian aggregation, expressed per capita of the overall population.\(^6\) If we assume that overall welfare is the summation of these subgroup welfares, we obtain

**Theorem 2.3 (Atkinson-Bourguignon).** For any \( f, g \in D_H(\alpha) \), \( f \gtrsim_{\text{SGL}} g \) holds if, and only if,

\[ \sum_{i=1}^H I(u_i, f_i) \geq \sum_{i=1}^H I(u_i, g_i) \quad \forall u \in \mathcal{U}_H \]

This theorem generalizes the result (for the GL ordering) noted in Remark 2.2, and demonstrates that the welfare ranking implied by the SGL dominance is unambiguous insofar as one is comfortable with the utilitarian preference aggregation. Theorem 2.3 is the main reason why the SGL ordering has been of interest to date.

### 3. Quasiconcave welfare functions and \( \gtrsim_{\text{SGL}} \)

We shall call any continuous and increasing function \( W: \mathbb{R}^H \rightarrow \mathbb{R} \) a social welfare function (SWF). For any \( (f, u) \in D_H(\alpha) \times \mathcal{U}_H \), a SWF determines the total welfare of the population as a certain aggregation of the subgroup welfares: \( W = W(I(u_1, f_1), \ldots, I(u_H, f_H)) \). The class of all SWFs is denoted by \( \mathcal{W}_H \). A linear SWF is said to be a Bergson-Samuelson SWF. A Bergson-Samuelson SWF that assigns equal weight to each subgroup is called a utilitarian SWF.

**Definition 3.1.** Let \( W \in \mathcal{W}_H \) and let \( \gtrsim \) be any preorder on \( D_H(\alpha) \). We say that \( \gtrsim \) is supported by \( W \) whenever, for all \( f, g \in D_H(\alpha) \), \( f \gtrsim g \) holds if, and only if,

\[ W(I(u_1, f_1), \ldots, I(u_H, f_H)) \geq W(I(u_1, g_1), \ldots, I(u_H, g_H)) \quad \forall u \in \mathcal{U}_H \]

Theorem 2.3 establishes that \( \gtrsim_{\text{SGL}} \) is supported by the utilitarian SWFs. Since the SGL ordering is an extension of the GL ordering, it is natural to ask if the SGL ordering has also the support of all symmetric and quasi-concave SWFs. Unfortunately, such a hope is unwarranted, and leads one to conclude that the determination of the welfare support of the SGL ordering is a non-trivial problem:

\(^5\)While \( I \) also depends on \( i \), we do not adopt a notation that makes this explicit to simplify the exposition. Given the assumption of fixed marginals, this convention does not cause any problems.

\(^6\)Since all individuals in group \( i \) have the same utility function, intragroup interpersonal utility comparisons do not pose a problem in this process (cf. Harsanyi, 1990, p. 131). On the other hand, assuming that subgroup welfares are summable entails comparisons of welfare differences between the subgroups.

\(^7\)What we mean by a SWF in this paper is thus a **subgroup welfare aggregator**.
Proposition 3.2. For any $H \geq 2$, the SGL ordering is not supported by all symmetric and quasi-concave SWFs.

Proof. This proposition is an immediate corollary of our main result, Theorem 4.3. The details of the proof are thus relegated to a later section.

What is then the preorder on $\mathcal{D}_H(\alpha)$ that is supported by all quasi-concave SWFs? The following theorem yields a characterization of this welfare ordering. Its proof is easy and omitted.

Proposition 3.3. Let $f, g \in \mathcal{D}_H(\alpha)$. The following statements are equivalent:

- (a) $f_i \succeq _{GL} g_i$ for all $i = 1, \ldots, H$.
- (b) $W(I(u_1, f_1), \ldots, I(u_H, f_H)) \succeq W(I(u_1, g_1), \ldots, I(u_H, g_H))$ for all $(u, W) \in \mathcal{U}_H \times \mathcal{W}_H$.
- $\sum \theta I(u_i, f_i) \succeq \sum \theta I(u_i, g_i)$ for all $(u, (\theta_1, \ldots, \theta_H)) \in \mathcal{U}_H \times \mathbb{R}^+_H$.

In passing, we note that the equivalence of (b) and (c) in Proposition 3.3 yields the following counterpart of Proposition 3.2:

Corollary 3.4. For any $H \geq 2$, the SGL ordering is not supported by all Bergson-Samuelson SWFs.

4. Needs-based welfare functions and $\succeq _{SGL}$

In this section we investigate those SWFs that are actually compatible with the SGL ordering. Let us begin with answering the following question: what kind of Bergson-Samuelson SWFs support the SGL ordering? The answer is quite intuitive. Since $\succeq _{SGL}$ is an ordering specifically designed to rank income distributions when needs differ, it is natural that it would seek the support of SWFs which put emphasis on these need differences. Put formally, we have

Theorem 4.1. For any $f, g \in \mathcal{D}_H(\alpha)$, $f \succeq _{SGL} g$ holds if, and only if,

$$\sum_{i=1}^H \theta_i I(u_i, f_i) \succeq \sum_{i=1}^H \theta_i I(u_i, g_i) \ \forall u \in \mathcal{U}_H \text{ and } \theta_1 \geq \cdots \geq \theta_H > 0$$

Proof. Sufficiency follows from Theorem 2.3. To see necessity, let $f \succeq _{SGL} g$ for some $f, g \in \mathcal{D}_H(\alpha)$ and take any $u \in \mathcal{U}_H$. Since $(\theta_1 u_1, \ldots, \theta_H u_H) \in \mathcal{U}_H$ provided that $\theta_1 \geq \cdots \geq \theta_H > 0$, Theorem 2.3 entails that $\sum \theta_i I(u_i, f_i) = \sum I(\theta_i u_i, f_i) \succeq \sum I(\theta_i u_i, g_i) = \sum \theta_i I(u_i, g_i)$.

We now extend this observation to a more general class of SWFs.

As we shall see, one can sharpen this observation as follows: For any $H \geq 2$, the SGL ordering is not compatible with the Rawlsian method of socially evaluating heterogeneous income distributions. (Here Rawlsianism can be defined with respect to either the worst off subgroup or the worst off individual in the society at large.)

Whether the neediest group should be treated the same way with the least needy one in social welfare evaluations is a question which cannot be a priori assumed away (Fine, 1985). Thus, here we do not restrict attention to only the symmetric SWFs.
Definition 4.2. Let $H \geq 2$. A SWF $W$ is said to be needs-based if, for any $k \in \{1, \ldots, H-1\}$,
\[
W(a_1, \ldots, a_H) \geq W(a_1, \ldots, a_{k-1}, a_k - \epsilon, a_{k+1} + \epsilon, a_{k-2}, \ldots, a_H)
\]
for all $\epsilon \in [0, a_k]$ and all $(a_1, \ldots, a_H) \in \mathbb{R}^H$. We denote the set of all needs-based SWFs by $\mathcal{W}_{nb}^H$.

Needs-based SWFs are those that would never record a decrease in overall welfare if a (cardinal) welfare transfer were made from the less needy to the needier. That is, a needs-based SWF has its marginal rate of substitution of the $i$th component for the $(i+1)$th component always greater than 1 in absolute value, $i = 1, \ldots, H-1$. One may argue that such asymmetric welfare functions correct for the subgroups which are in a relatively disadvantageous situation in the population, and thus that they are normatively more appealing in the present framework (Dardanoni, 1993).

A Bergson-Samuelson SWF is needs-based if, and only if, $u \equiv u$. Therefore, it is natural to ask if Theorem 4.1 can be extended to the domain of all needs-based SWFs. Our main result answers this question in the affirmative:

Theorem 4.3. A SWF $W \in \mathcal{W}_{nb}^H$ supports the SGL ordering if, and only if, it is needs-based.

The proof is given in Appendix A.

Theorem 4.3 shows that the SGL ordering is supported by all and only needs-based SWFs: the largest class of SWFs that support $\succsim_{SGL}$ is $\mathcal{W}_{nb}^H$. Our main result, therefore, completely identifies the ethical properties of SGL dominance with regard to the aggregation of subgroup welfare.

In passing, we note that Theorem 4.3 entails Proposition 3.2 as an immediate corollary. For, the Rawlsian SWF $W:(a_1, \ldots, a_H) \rightarrow \min a_i$ is a symmetric and quasi-concave SWF which is not needs-based.

5. Conclusion

This paper has examined the welfare support of the SGL ordering with respect to the aggregation of the well-being levels of the constituent subgroups. The findings of Section 3 have determined what the SGL ordering does not achieve with respect to subgroup welfare aggregation: If one wishes to use an arbitrary symmetric and quasi-concave SWF in this aggregation, then SGL dominance is not the right welfare ordering to use. Section 4, on the other hand, has identified what SGL dominance is actually capable of in this regard: If one wishes to use an arbitrary needs-based SWF, then $\succsim_{SGL}$ is not only the right ordering but is actually the only ordering to use. Thus, as long as one believes that $\mathcal{W}_{nb}^H$ includes all the ‘sensible’ subgroup welfare aggregation methods, $\succsim_{SGL}$ turns out to be a very appealing welfare ordering. In conclusion, we note that such a position is of course defended by numerous welfare economists and philosophers. For instance, Letwin (1983), p. 8, states that “inasmuch as people are unequal, it is rational to presume that they ought to be treated unequally—which might mean larger shares for the needy . . . ”. (See also Fine, 1985, and Dardanoni, 1993.)

\textsuperscript{10} An extreme member of $\mathcal{W}_{nb}^H$ is the SWF of the form $W:(a_1, \ldots, a_H) \rightarrow a_i$ for all $(a_1, \ldots, a_H) \in \mathbb{R}^H$. Jenkins and Lambert (1993) refer to this SWF as quasi-‘Rawlsian’. For $H = 2$, $W(a,b) = a + \varphi(b)$ is needs-based for all $(a,b) \in \mathbb{R}^2$, where $\varphi \in C^1$ and $\varphi' < 1$. For instance, $W:(a,b) \rightarrow a + \arctan(b)$ is a needs-based SWF.
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Appendix A. Proof of Theorem 4.3

 Sufficiency: Let \( f, g \in \mathcal{D}_n(\alpha) \). We wish to show that \( f \succeq_{SGL} g \) iff \( W(I(u_1, f_1), \ldots, I(u_H, f_H)) \geq W(I(u_1, g_1), \ldots, I(u_H, g_H)) \) for all \( (u, W) \in \mathcal{U}_H \times \mathcal{W}^{nb} \). The ‘if’ part of this statement follows from Theorem 2.3. To see the ‘only if’ part, let \( f \succeq_{SGL} g \), and pick any \( W \in \mathcal{W}^{nb} \) and \( u \in \mathcal{U}_H \). By Definition 2.1 and Theorem 2.3, \( \sum_{i=1}^{k} (I(u_i, f_i) - I(u_i, g_i)) \geq 0 \) for all \( k = 1, \ldots, H \). Therefore, applying the needs-basedness property successively, we obtain

\[
W(I(u_1, f_1), \ldots, I(u_H, f_H)) \geq W(I(u_1, f_1) - (I(u_1, f_1) - I(u_1, g_1)), I(u_2, f_2)
+ (I(u_1, f_1) - I(u_1, g_1)), I(u_2, f_2), \ldots, I(u_H, f_H))
\]
\[
= W(I(u_1, g_1), I(u_2, f_2) + (I(u_1, f_1) - I(u_1, g_1)), I(u_3, f_3), \ldots, I(u_H, f_H))
\]
\[
\geq \cdots
\]
\[
\geq W(I(u_1, g_1), \ldots, I(u_{k-1}, g_{k-1}), (I(u_k, f_k) + \sum_{i=1}^{k-1} (I(u_i, f_i) - I(u_i, g_i)))
- \sum_{i=1}^{k} (I(u_i, f_i) - I(u_i, g_i)), I(u_{k+1}, f_{k+1}) + \sum_{i=1}^{k} (I(u_i, f_i))
- I(u_k, g_i), I(u_{k+2}, f_{k+2}), \ldots, I(u_H, f_H))
\]
\[
= W(I(u_1, g_1), \ldots, I(u_k, g_k), I(u_{k+1}, f_{k+1}) + \sum_{i=1}^{k} (I(u_i, f_i))
- I(u_k, g_i), I(u_{k+2}, f_{k+2}), \ldots, I(u_H, f_H)) \geq \cdots
\]
\[
\geq W(I(u_1, g_1), \ldots, I(u_{H-1}, g_{H-1}), (I(u_H, f_H) + \sum_{i=1}^{H-1} (I(u_i, f_i) - I(u_i, g_i)))
\]

But by Theorem 2.3, \( \sum (I(u_i, f_i) - I(u_i, g_i)) \geq 0 \), i.e.,

\[
I(u_{H}, g_{H}) - I(u_{H}, f_{H}) \leq \sum_{i=1}^{H-1} (I(u_i, f_i) - I(u_i, g_i))
\]

Therefore, since \( W \) is increasing by hypothesis,

\[
W(I(u_1, g_1), \ldots, I(u_{H-1}, g_{H-1}), I(u_H, f_H) + \sum_{i=1}^{H-1} (I(u_i, f_i) - I(u_i, g_i)) \geq
W(I(u_1, g_1), \ldots, I(u_{H-1}, g_{H-1}), I(u_H, f_H) + (I(u_H, g_H) - I(u_H, f_H)) + W(I(u_1, g_1), \ldots, I(u_{H-1}, g_{H-1}), I(u_H, g_H))
\]

and we obtain \( W(I(u_1, f_1), \ldots, I(u_H, f_H)) \geq W(I(u_1, g_1), \ldots, I(u_H, g_H)) \).
Therefore, Eqs. (2) and (3) yield that $I$ and $11$ as is sought. SGL (1), we have
$$W(t; a, b, e) > W(a, b).$$ Furthermore, by continuity of $W$, there must exist $\delta \in (0, \epsilon)$ such that
$$W(a - \epsilon, b + \delta) > W(a, b)$$ (1)

Define $v_1(t; \alpha, \xi) = \sigma t + \xi$ and $v_2(t; \beta, \tau) = \beta t + \tau$ for all $t \in [0, \alpha]$, where $\alpha, \xi, \beta, \tau \in \mathbb{R}$, and note that, for any $\xi, \tau$ and $\alpha > \beta$, we have $(v_1(; \alpha, \xi), v_2(; \beta, \tau)) \in \mathcal{U}_2$. Define
$$f_1(t) = g_2(t) = \frac{1}{\alpha}, \quad 0 \leq t \leq \alpha \quad \text{and} \quad g_1(t) = f_2(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & 1 < t \leq \alpha \end{cases}$$

We have
$$\int_{0}^{\alpha} \int_{0}^{y} (g_1(t) - f_1(t)) \, dx \, dt = \begin{cases} \frac{(\alpha - 1)}{2\alpha} y^2, & 0 \leq y \leq 1 \\ \ell(y), & 1 < y \leq \alpha \end{cases}$$

where $\ell(y) = y - (y^2 + \alpha)/2\alpha$ for any $y \in [0, \alpha]$. One can easily check that $\ell(y) > 0$ if and only if $\alpha(1 - \sqrt{\alpha - 1}/\alpha) < y \leq \alpha$. Thus, since $\alpha > 1$ by hypothesis, $\ell(y) > 0$, and we have $\int_{0}^{s} \int_{0}^{s} g_1(t) \, dx \, dt > \int_{0}^{s} \int_{0}^{s} f_1(t) \, dx \, dt$ for all $y \in [0, \alpha]$. Furthermore, $f_1 + f_2 = g_1 + g_2$ so that $f > g$. On the other hand, we have
$$I(v_1(; \alpha, \xi), f_1) = \frac{\alpha \xi}{2} + \xi \quad \text{and} \quad I(v_2(; \beta, \tau), f_2) = \frac{\beta \tau}{2} + \tau$$ (2)

and
$$I(v_1(; \alpha, \xi), g_1) = \frac{\alpha \xi}{2} + \xi \quad \text{and} \quad I(v_2(; \beta, \tau), g_2) = \frac{\beta \alpha}{2} + \tau$$ (3)

Therefore, Eqs. (2) and (3) yield that $I(u_1, f_1) = a$, $I(u_2, f_2) = b$, $I(u_1, g_1) = a - \epsilon$ and $I(u_2, g_2) = b + \delta$, where
$$u_1(t) = v_1(t; \alpha, \xi, a - \frac{\alpha \epsilon}{\alpha - 1}) \quad \text{and} \quad u_2(t) = v_2(t; \beta, \tau, b - \frac{\delta}{\alpha - 1})$$

for all $t \in [0, \alpha]$. (Notice that $\delta < \epsilon$ implies that $u'_1 > u'_2$ so that $(u_1, u_2) \in \mathcal{U}_2$.) Consequently, by Eq. (1), we have $W(I(u_1, f_1), I(u_2, f_2)) < W(I(u_1, g_1), I(u_2, g_2))$. This establishes that $W$ does not support $\simeq_{\text{SGL}}$ as is sought.

\(^{11}\)Of course setting $p_1 = p_2 = 1$ is identical to assuming that $p_1 = p_2 = 1/2$, for the present purposes.
References